The Differential Geometry of the General Helix as Applied to Mechanical Springs

Yuyi Lin
Research Assistant.

Albert P. Pisano
Associate Professor.
Assoc. Mem. ASME

Mechanical Engineering Department,
University of California,
Berkeley, CA 94720

In order to improve the performance of helical springs, such as increasing the fatigue life and suppressing resonance, variable pitch angle and variable helix radius may be incorporated into the helical spring geometry. Employing the tool of differential geometry, new and complete formulae of curvature, torsion, and spring force are derived. It is shown that these formulae are more general and accurate than Kelvin's curvature and torsion formulae, than commonly used force formulae (Wahl, 1963). Possible simplifications to the complete formulae and the corresponding errors introduced are both discussed and compared with experimental data.

Introduction

Modern study and research concerning helical springs dates back at least a century (Thomson and Tait, 1883) and some dated assumptions about helical spring geometry, no matter how impractical, are still being applied in modern day analyses. For example, helical springs usually are assumed to have constant pitch angle throughout the spring. However, for stably seating the spring between two parallel end surfaces in practical applications, the pitch angle at the two ends of spring is usually made smaller in order to "close" the coils. Hence, to assume constant pitch angle is to assume an infinite helix length, or a spring without stable and parallel seating ends, and to ignore all end effects. Similarly, some fundamental formulae concerning the geometry of the helix, based on historically made assumptions, are still applied without awareness of the error introduced. For instance, the formulae dating from early work (Thomson and Tait, 1883) expressing curvature, κ, and torsion, τ, in terms of pitch angle, p, and helix radius, r,

$$\kappa = \frac{\cos^2 p}{r} \tag{1}$$

$$\tau = \frac{\sin p \cos p}{r} \tag{2}$$

are still widely used (Love, 1927; Wahl, 1963; Wittrick, 1966; Stokes, 1974; Pearson, 1982; Velinsky, 1987).

In searching for higher performance, springs with large variations from traditional geometries have been manufactured and tested (Fig. 1). However, there is no accurate mathematical description of this new type of spring in the open literature. The purpose of this paper is to utilize differential geometry to derive more accurate and more general formulae for curvature, torsion, and spring force and apply these to the helical spring. The errors introduced using old or simplified formulae are discussed and compared both to the formulae derived herein and to experimental data. It is hoped this paper can contribute an analytical tool to the study of helical spring dynamics, and offer some constructive ideas for a new class of designs of helical springs.

Description of the General Helical Spring

It is a common assumption (Fig. 2), that the helical spring

Fig. 1 Helical springs with large variations in radius and pitch angle

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for presentation at the Joint ASCE/ASME Applied Mechanics, Biomechanics, and Fluids Engineering Conference, San Diego, CA, July 9 to 12, 1989.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, January 1, 1988; final revision, April 29, 1988.

Paper No. 89-APM-17.

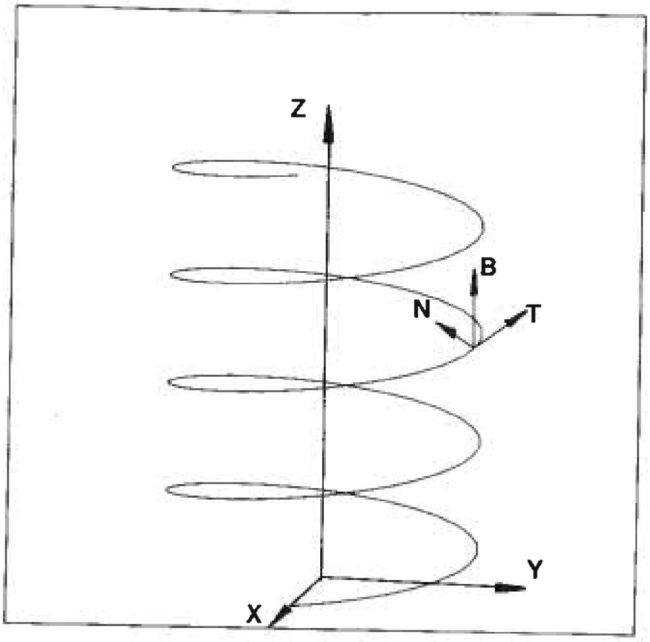

Fig. 2 Circular helix and local-global coordinate systems

can be described to sufficient accuracy by a circular helix (Eisenhart, 1940), which can be considered the trace of a curve, X, in R^3 space, such that:

$$X{:}I \rightarrow R^3 \tag{3}$$

where a parametrization of X is such:

$$X(t) = (r \cos at, r \sin at, bt) \tag{4}$$

where the parameter $t \epsilon I$, I is a subset of R, and r, a, and b are constants.

For the idealized helix treated in general textbooks (Darboux, 1887; Vaisman, 1984), this description is clear and simple. However, even if a commonly used helical spring (where end effects are small due to design) is described in this fashion, the error in predicted spring force for a given, imposed spring displacement can be as much as 5 percent in simple static compression (Lin and Pisano, 1987). An even greater error is introduced when the pitch angle is assumed to be constant throughout the spring in the analysis, but the physical implementation of the spring includes not only closed coils at each end, but a deliberately varying pitch angle in the body of the spring. Since spring technology has been moving toward variable pitch angle designs, an accurate description of pitch angle is both useful and necessary. A precise definition of pitch angle follows:

Definition: PITCH ANGLE, p, is the complement angle of the angle between tangent T and the Z axis. Alternatively, pitch angle p is the angle between the rectifying plane of the intrinsic coordinate and X-Y plane of the global coordinate system.

The tangent plane (Fig. 2) is the plane spanned by the T and B axes, the rectifying plane is the plane spanned by the T and N axes, and the intrinsic coordinate is the Frenet Triad (Do Carmo, 1976).

It is easy to show that the pitch angle for the circular helix is constant (Do Carmo, 1976). However, springs in practical applications have a small, nearly zero pitch angle at each end. In addition, a varying helix radius may help improve spring dynamic performance by de-tuning the spring resonance. We use the term general helix to describe the varying pitch angle and varying radius helix.

Definition: GENERAL HELIX is the trace of a curve

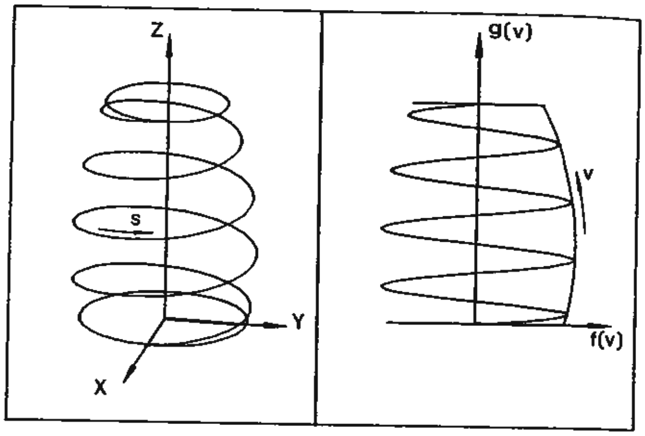

Fig. 3 Describing a general helix

$\alpha(s){:}I \rightarrow R^3$, such that α can be embedded on a surface of revolution, with pitch angle a non-negative function of the parameter, s.

Compared with the circular helix, equation (4), a parametrization of a general helix can be:

$$X(s) = (r(s)\cos \theta(s), r(s)\sin \theta(s), h(s)) \tag{5}$$

The surface of revolution mentioned in the above definition can be parametrized as:

$$Y(u,v) = (f(v)\cos u, f(v)\sin u, g(v)) \tag{6}$$

where f, g are known functions of v, a parameter preferably taken as the arc length, s, because the arc length of a regular curve is an invariant with respect to transformations of the parameter used to define the curve (Stoker, 1969). This is reasonable since in practice spring wire can usually be assumed to be inextensible. A curve, shown in Fig. 3, on $Y(u, v)$ can be determined by either of the following relations:

$$u = u(t), \qquad v = v(t) \tag{7a}$$

or

$$u = u(v) \tag{7b}$$

If equation (7a) is chosen and t is the arc length parameter, then the general helix is described by equation (5). If equation (7b) is chosen, then the curve has the parametrization:

$$X(v) = (f(v)\cos u(v), f(v)\sin(v), g(v)) \tag{8}$$

Although equation (5) and equation (8) have similar forms, parameters v and s have different meanings. Intuitively, it is seen that v is arc length of the helix on the Y-Z plane, and s is the actual wire length (Fig. 3). They can be related by:

$$ds^2 = [E(u_v')^2 + G]dv^2 \tag{9}$$

where E and G are coefficients of first fundamental form (Do Carmo, 1976) of the surface $Y(u,v)$. Using the first fundamental form here enables us to measure arc length in terms of surface coordinates without referring back to the ambient R^3 space where the surface lies (Do Carmo, 1976). In the discussion to follow, the general helix will be represented by equation (5) for convenience.

New Formulae for Curvature and Torsion

Many researchers and engineers have been using the formulae for curvature, κ, and torsion, τ, from the Treatise on Natural Philosophy (Lord Kelvin and Tait, 1883). The formulae state:

$$\kappa = \frac{\cos^2 p}{r} \tag{1}$$

and

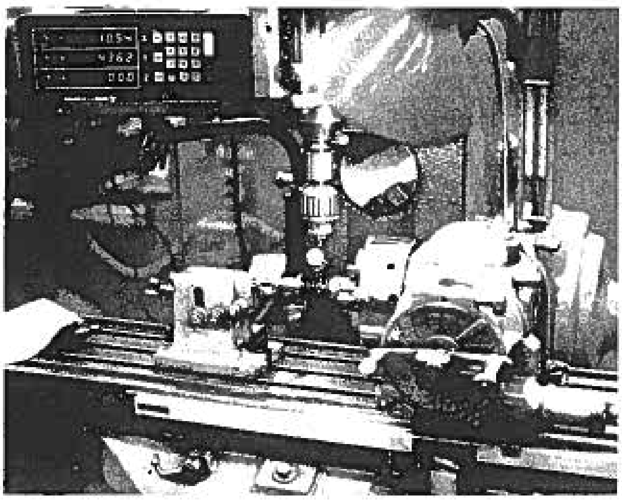

Fig. 4 Measuring radii on a milling machine

$$\tau = \frac{\sin p \cos p}{r} \tag{2}$$

where p is the pitch angle, and r the radius, of the helix. These formulae were derived intuitively (Thomson and Tait, 1883).

Many researchers use the above formulae for a constant radius helix spring even though they realize the pitch angle is a function of helix length. Note that even if the radius is still constant, varying pitch has made the helix not a circular helix in the strict sense of Eisenhart (1940). In addition, if a dynamic spring deflection is under consideration, the helix radius and pitch angle will not be constant in time even for a circular helix spring. We therefore derive these formulae for the curvature and torsion of the general, as opposed to circular, helix.

Derivation: Let a general helix parametrized by arc length be:

$$X(s) = (r \cos \theta,\ r \sin \theta,\ h) \tag{10}$$

where helix radius, r, polar angle, θ, and local helix height, h, are functions of arc length, s. Then derivatives with respect to arc length, s, are:

$$\begin{aligned} X'(s) &= \mathbf{T}(s) \\ &= (r' \cos \theta - \theta' r \sin \theta,\ r' \sin \theta + \theta' r \cos \theta,\ h') \end{aligned} \tag{11}$$

where $\mathbf{T}(s)$ is unit tangent vector of $X(s)$, since s is the arc length parameter.

$$\begin{aligned} X''(s) &= \mathbf{T}'(s) \\ &= (r'' \cos \theta - 2\theta' r' \sin \theta - r\theta'' \sin \theta - r\theta'^2 \cos \theta, \\ &\quad r'' \sin \theta + 2\theta' r' \cos \theta - \theta'^2 r \sin \theta + r\theta'' \cos \theta,\ h'') \end{aligned} \tag{12}$$

From Frenet-Serret formulae (Eisenhart, 1940), the curvature is the magnitude of $\mathbf{T}'(s)$:

$$\mathbf{T}'(s) = \kappa \mathbf{N}(s) \tag{13}$$

and since $\mathbf{N}(s)$ is a unit vector, taking inner product of $\mathbf{T}'(s)$, we obtain the full expression for curvature:

$$\begin{aligned} \kappa^2 &= (X'', X'') \\ &= r^2\theta'^4 + 4r'^2\theta'^2 + r^2\theta''^2 - 2r\theta'^2 r'' + 4rr'\theta'\theta'' + r''^2 + h''^2 \end{aligned} \tag{14}$$

Note that in contrast, the common analysis of analytical geometry gives:

$$\begin{aligned} \cos^2 p &= \frac{r'^2 + r^2\theta'^2}{|X'|^2} \\ &= r'^2 + r^2\theta'^2 \end{aligned} \tag{15}$$

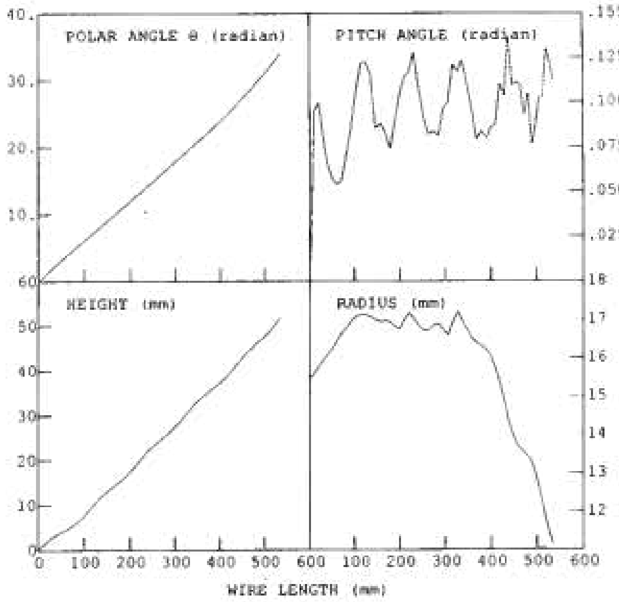

Fig. 5 Description of sample spring "P"

and then from Kelvin's curvature formula,

$$\begin{aligned} \kappa^2 &= \left(\frac{\cos^2 p}{r}\right)^2 \\ &= r^2\theta'^4 + 2r'^2\theta'^2 + \frac{r'^4}{r^2} \end{aligned} \tag{16}$$

It is seen from equation (14) that κ is a function of r'' and h'' in general, and that this is not true in Kelvin's formula equation (16). In some special cases, for example, the circular helix (equation (4)) with constant radius ($r' = 0$), linear height ($h'' = 0$), and linear polar angle ($\theta'' = 0$), the new curvature expression is identical to the old formula.

Example: In Fig. 5, measured and smoothed data for a particular helical spring is shown. The spring has constant radius, and in the middle portion the pitch angle is constant, therefore it is locally a circular helix. It is seen from the numerical results shown in Table A (Case 1) that the Kelvin curvature formula and the complete curvature formula give the same result. For the same spring, at one end (Case 2, $s = 46.35$ mm) where pitch angle is varying, the curvature calculated by the Kelvin formula has an error of four parts in one hundred thousand.

The above example shows if the spring has only a small variation in pitch angle p, but no variation in radius r, the curvatures calculated from the complete formula and from the old formula are very close. However, sometimes this difference can be considerable, as shown later in this section.

Example: One of the helical springs shown in Fig. 1 has been measured for pitch angle, radius, and height as shown in Fig. 6. The measurement of variable radius is done on a milling machine as shown in Fig. 5. Note these results have been parametrized by arc length, s. At $s = 30.05$ mm (Case 3), curvature calculated from the Kelvin formula has an error of $+0.153$ percent with respect to experimental data.

The derivation of the torsion formula for the general helix is lengthy, and an outline of this derivation is given in the Appendix. It can be seen from the derivation that the Kelvin formula of τ depends on r, h, and θ only up to first derivatives with respect to arc length. In general, however, τ will also depend on second and third derivatives of r, h, and θ. Unless all these higher order derivatives vanish, a calculation of τ from the Kelvin formula will contain errors. The magnitude of the error depends on the magnitude of higher derivatives of r, h,

Table A: Comparison of curvature and torsion using different formulae

Case	1:spr.P	2:spr.P	3:spr.F	4:spr.F
s(mm)	120.66	46.35	30.05	114.76
κ_{old}(1/mm)	0.067245	0.067378	0.061595	0.057748
κ_{new}(1/mm)	0.067245	0.067381	0.061501	0.057808
error(%)	0.000	−0.004	0.153	−0.104
τ_{old} (1/mm)	0.007253	0.006612	0.003811	0.006770
τ_{new}(1/mm)	0.007253	0.006485	0.004011	0.003667
error(%)	0.000	1.961	−4.984	84.638
ϵ(deg.)	0.000	0.479	−0.799	−1.045

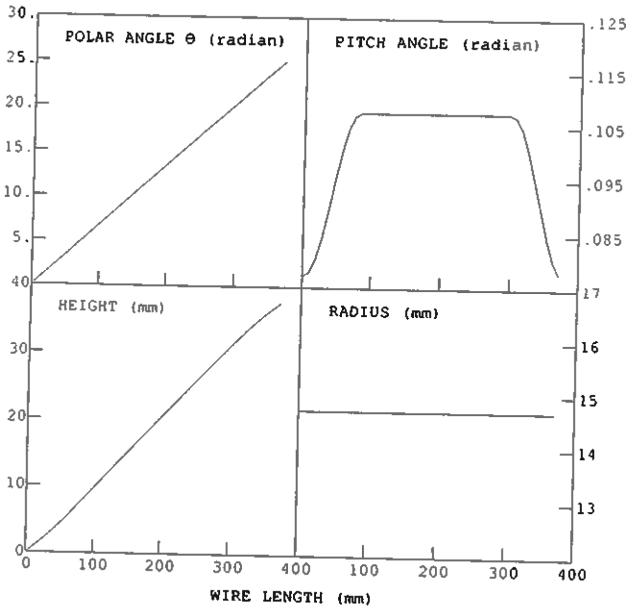

Fig. 6 Description of sample spring "F"

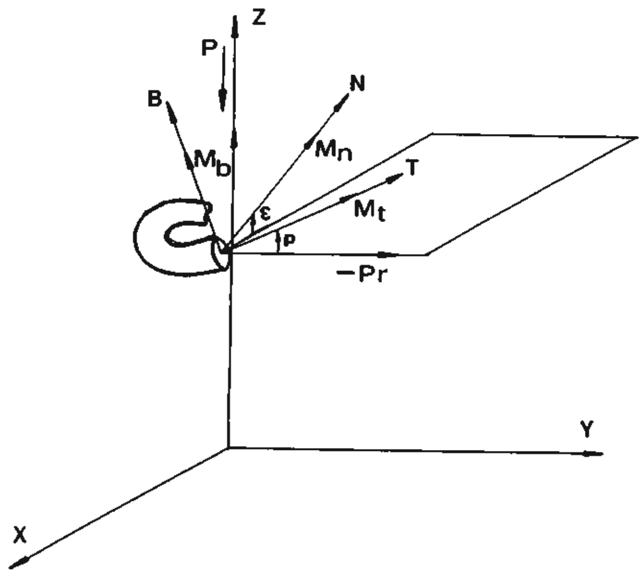

Fig. 7 Force and moment diagram for spring element

and θ of the specific helix. For more examples comparing results computed from old and new formulae, please refer to Table A. To minimize the effect of measurement error, the numerical results presented in Table A are calculated from third-order polynomial approximations of measured spring curve. The coefficients of these polynomials are obtained by least square curve fit. Note these cases are intended only to give some approximation of possible differences and they do not represent the largest possible differences. Nevertheless, in Case 4, the error in τ has calculated to be as great as +84.6 percent.

Change in the Direction of Normal Vector N

In an earlier paper (Lin and Pisano, 1987), and other work (Wahl, 1963), the normal, N, of the intrinsic coordinate system has always been considered parallel to X-Y plane of the global coordinate, for the purpose of calculating spring force and moment (Fig. 7). When a force, P, is applied to one end of the spring, at any point, q, a moment,

$$M_h = -\vec{r}\times\vec{P} \tag{17}$$

is produced. M_h will have to be balanced by resultant moments due to the stress in the spring material at point q. M_h can be decomposed into M_b and m_t, as shown in Fig. 7, which must be balanced by changes in curvature, κ, and torsion, τ. This kind of analysis has assumed that the normal, N, is parallel to the X-Y plane. It can be shown that the normal, N, is not parallel to the X-Y plane for the general helix.

Derivation: Let a parametrization of a general helix be:

$$\mathbf{X}(s) = (r\cos\theta,\ r\sin\theta,\ h) \tag{18}$$

where r,θ,h are functions of arc length, s. Then,

$$\begin{aligned} \mathbf{T}'(s) &= \kappa\,\mathbf{N} \\ &= (r''\cos\theta - 2\theta' r'\sin\theta - \theta'^2 r\cos\theta, \\ &\qquad r''\sin\theta + 2r'\theta'\cos\theta - \theta'^2 r\sin\theta,\ h'') \end{aligned} \tag{19}$$

$$\begin{aligned} \mathbf{N}(s) &= \frac{\mathbf{T}'}{|\mathbf{T}'|} \\ &= \frac{\mathbf{T}'}{\sqrt{r''^2 + 4\theta'^2 r'^2 + \theta'^4 r'^2 - 2\theta'^2 r'' r' + h''^2}} \\ &= \frac{\mathbf{T}'}{\kappa} \end{aligned} \tag{20}$$

N would be parallel to the X-Y plane, if and only if $h''=0$, i.e., the z coordinate function, $h(s)$, of the helix were a linear function of arc length s. Otherwise N is not parallel to the X-Y plane.

We can conclude from the above proof, that when $h''\neq 0$ the calculated values of M_b and M_t must be corrected by the cosine of the angle, ϵ, where ϵ is found from:

$$\epsilon = \sin^{-1}\left(\frac{h''}{\sqrt{r''^2 + 4\theta'^2 r'^2 + \theta'^4 r^2 - 2\theta'^2 r'' + h''^2}}\right) \tag{21}$$

and a new component of torque, M_n, in the direction of N will have to be balanced by additional stress in the spring helix. In a previous paper (Lin and Pisano, 1987), this balancing torque is called the bending moment in the T-B plane. To have a quantitative sense about the corrective angle, ϵ, consider two examples.

Example: An automotive valve spring has measured and smoothed initial pitch angle distribution data as shown in Fig. 5. Assume at 5 mm compression it has constant radius $r=0.0147$m. Since the measurement is made on nodes evenly distributed around a circle, the spring is conveniently represented by:

$$X(\theta)=(r\cos\theta,\ r\sin\theta,\ h(\theta))$$

However, it is more convenient in a later computation to reparametrize the spring by arc length:

$$X(s)=(r\cos\theta(s),\ r\sin\theta(s),\ h(\theta(s)))$$

with θ and s related by the mapping:

$$s(\theta)=\int_0^\theta |X'(\theta)|d\theta$$

$$=\int_0^\theta \sqrt{r^2+h'^2_\theta}\,d\theta$$

At arc length equal to 46.35 mm of the spring (Case 2 in Table A),

$$\epsilon\approx 0.5^\circ$$

Example: In Table A, Case 3, the spring has variable pitch angle and radius (shown in Fig. 6), and at arc length, s, equal to 114.76 mm, $\epsilon=-1.045$ deg.

New Formulae for Calculating Spring Forces

As a result of new formulae for calculating curvature, κ, torsion, τ, and the inclusion of a new strain energy term, a new spring force computation formula has been derived that differs from common formulae (Wahl, 1963) currently in use. Referring to Fig. 7, when a force P in the negative Z direction is applied, the internal force at any cross-section of the spring produced by strain can be decomposed into a force, F, in the positive Z direction, passing through the center of the spring wire, and into a moment, M, about Z axis, described below as:

$$F=\frac{GJ\cos p}{r}(\Delta\tau)-\frac{EI\sin p}{r}((\Delta\kappa)\cos\epsilon+(\Delta z'')\sin\epsilon) \quad (22)$$

$$M=GJ\sin p(\Delta\tau)+EI\cos p((\Delta\kappa)\cos\epsilon+(\Delta z'')\sin\epsilon) \quad (23)$$

If an accurate evaluation of the spring force is desired, the above formula is recommended. It should be noted that although the above formulae are analytical and exact, it turns out to be difficult to obtain exact values for changes in curvature and torsion, either numerically or experimentally. For the case where the spring has a constant helix radius and varying pitch angle, an experimentally-verified, simplified formula, which is repeated here for completeness, can be used to obtain very good results (Lin and Pisano, 1987):

$$\Delta p=\text{constant with respect to arc length} \quad (24)$$

where Δp is the change of pitch angle corresponding to specified spring deflection. After Δp is solved, changes in curvature and torsion are calculated from Kelvin formulae using new pitch angle distribution.

Example: If static spring force for spring "P", which is described in Table A and Fig. 5, is computed by simple formula (Shigley, 1983), maximum error between computed force and experimentally measured force is about 40 Newton (5 percent). This large error is resulted from a considerably large varying pitch angle. If approximate formula, equation 24, is applied to calculate change of curvature, $\Delta\tau$, and only first term in equation 22 is utilized to calculate the spring force, maximum error is about 9 Newton (1 percent). The third term in equation 22 has no significant contibution, because in this case $\Delta z''$ and ϵ are almost zero along the helix.

Conclusions

The more general and accurate mathematical model of a helical spring is general helix rather than circular helix. The most significant differences between the general and the circular helices are varying pitch angle and varying radius, and consequently the historically used curvature and torsion formulae have been shown to introduce errors that vary depending on application and specific configuration. A word of caution is that for realistic helical spring models, especially when spring dynamics is concerned, no helical spring is an exact circular helix. New formulae for curvature, torsion, and spring force have been derived to handle this case and are recommended where accuracy is more important than computational effort. With the availability of accurate helix models, it is now possible to take advantage of the complex shape of general helix in spring design, so as to obtain higher and variable natural frequencies and subsequently longer fatigue life.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant Nos. MEA-8404822 and DMC-8451199 (a Presidential Young Investigator Award). Other support has been provided by the General Motors Corporation, CPC Group, under research contract No. BGMC-DD138898. The assistance of Associate Spring Company in the form of providing sample springs for this investigation is gratefully acknowledged. The first author wishes to express his appreciation to Prof. Charles Pugh, of the Department of Mathematics of UC Berkeley, for his encouragement in writing this paper. The authors also wish to acknowledge the support of the Department of Mechanical Engineering, University of California, Berkeley, in the providing of computer and laboratory resources.

References

Darboux, G., 1887, *Lecons sur la Theorie Generale des Surfaces*, Gauthier-Villars, Imprimeur-Libraire, Paris, p. 11.

Do Carmo, M. P., 1976, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, pp. 19–22.

Costello, G. A., 1975, "Radial Expansion of Impacted Helical Springs," ASME JOURNAL OF APPLIED MECHANICS, Vol. 42, pp. 789–792.

Eisenhart, L. R., 1940, *An Introduction to Differential Geometry*, Princeton University Press, pp. 14–15.

Kato, M., 1974, "Study on Dynamic Properties of a Varying-Pitch Helical Compression Spring," *Bulletin of the JSME*, Vol. 17, No. 110, pp. 1015–1022.

Lin, Y., and Pisano, A. P., 1987, "General Dynamic Equations of Helical Springs with Static Solution and Experimental Verification," ASME JOURNAL OF APPLIED MECHANICS, Vol. 54, pp. 910–917.

Love, A. E. H., 1927, *A Treatise on the Mathematical Theory of Elasticity*, 4th Ed., Dover, New York.

Pearson, D., 1982, "The Transfer Matrix Method for the Vibration of Compressed Helical Springs," *Journal of Mechanical Engineering Science*, Vol. 24, No. 4, pp. 163–171.

Phillips, J. W., and Costello, G. A., 1972, "Large Deflections of Impacted Helical Springs," *Journal of the Acoustical Society of America*, Vol. 51, No. 3, pp. 967–973.

Pisano, A. P., and Freudenstein, F., 1983, "An Experimental and Analytical Investigation of the Dynamic Response of a High-Speed Cam Follower System, Part 2: A Combined, Lumped/Distributed Parameter Dynamic Model," ASME *Journal of Mechanisms, Transmissions and Automation in Design*, Vol. 105, pp. 699–704.

Sayre, M. F., and De Forest, A. V., 1939, "New Spring Formulas and New Materials for Precision Spring Scales," *Transactions ASME*, Vol. 58, pp. 379–387.

Shigley, J. E., and Mithcell, L. D., 1983, *Mechanical Engineering Design*, 4th Ed., McGraw-Hill, New York, p. 447.

Stoker, J. J., 1969, *Differential Geometry*, John Wiley, New York, pp. 14–21.

Stokes, V. K., 1974, "On the Dynamic Radial Expansion of Helical Springs due to Longitudinal Impact," *Journal of Sound and Vibration*, Vol. 35, pp. 77–99.

Thomson, W., and Tait, P. G., 1883, *Treatise on Natural Philosophy*, 2nd Ed., Oxford, pp. 136–144.

Vaisman, I., 1984, *A First Course In Differential Geometry*, Marcel Dekker, New York, pp. 72–73.

Velinsky, S. A., 1987, "Design and Mechanics of Multi-Lay Wire Strands," *Advances in Design Automation, Proceedings of ASME 13th Annual Design Automation Conference*, Vol. 1, pp. 221–231.

Wahl, A. M., 1963, *Mechanical Springs*, Penton Publishing Co., Cleveland, OH.

Wittrick, W. H., 1966, "On Elastic Wave Propagation in Helical Springs," *International Journal of Mechanical Science*, Vol. 8, pp. 25–47.

APPENDIX

Torsion of General Helix

Assume a general helix parametrized by arc length is:

$$X(s) = (r\cos\theta,\ r\sin\theta,\ h) \tag{A-1}$$

where r, θ, and h are functions of arc length s. From the Frenet-Serret formula of differential geometry,

$$\mathbf{N}' = -\kappa\mathbf{T} - \tau B \tag{A-2}$$

$$(\mathbf{N}', \mathbf{N}') = \kappa^2 + \tau^2 \tag{A-3}$$

$$\tau^2 = \kappa^2 - (\mathbf{N}', \mathbf{N}') \tag{A-4}$$

Since

$$\mathbf{N} = \frac{\mathbf{T}'}{\kappa} \tag{A-5}$$

$$\mathbf{N}' = \frac{\mathbf{T}''\kappa - \kappa'\mathbf{T}'}{\kappa^2} \tag{A-6}$$

where

$$\begin{aligned}\mathbf{T}' &= \mathrm{X}'' \\ &= (r''\cos\theta - 2r'\theta'\sin\theta - r\theta''\sin\theta - r\theta'^2\cos\theta, \\ &\quad r''\sin\theta + 2r'\theta'\cos\theta + r\theta''\cos\theta - r\theta'^2\sin\theta,\ h'')\end{aligned} \tag{A-7}$$

$$\begin{aligned}\mathbf{T}'' = (&r'''\cos\theta - 3r''\theta'\sin\theta - 3r'\theta''\sin\theta - 3r'\theta'^2\cos\theta \\ &- r\theta'''\sin\theta - 3r\theta'\theta''\cos\theta + r\theta'^3\sin\theta, \\ &r'''\sin\theta + 3r''\theta'\cos\theta + 3r'\theta''\cos\theta - 3r'\theta'^2\sin\theta \\ &+ \theta''' r\cos\theta - 3r\theta'\theta''\sin\theta - r\theta'^3\cos\theta,\ h''')\end{aligned} \tag{A-8}$$

$$\begin{aligned}\kappa' = \frac{1}{2\kappa}(&2h''h''' + 2r''r''' + 6r'r''\theta'^2 + 12r'^2\theta'\theta'' \\ &+ 6rr'\theta''^2 + 2r^2\theta''\theta''' \\ &+ 2rr'\theta'^4 + 4r^2\theta'^3\theta'' - 2r\theta'^2 r''' + 4rr'\theta'\theta''')\end{aligned} \tag{A-9}$$

Then finally,

$$\tau = \sqrt{\kappa^2 - \frac{1}{\kappa^4}(\kappa^2(\mathbf{T}'',\mathbf{T}'') + \kappa'^2(\mathbf{T}',\mathbf{T}') - 2\kappa\kappa'(\mathbf{T}',\mathbf{T}''))} \tag{A-10}$$