01325 Mathematics 4, Spring 2013

Week no. 12

Theory: Wednesday May 8 we complete the treatment of wavelets (page 169 – 170) and discuss Sections 10.1–10.3.

Exercises for Wednesday May 8: 10.2 (i), 10.1 (only proof of Theorem 10.1.3 (iii), see comment below), 10.8 (hint below), 10.11, 10.10

Comment to Exercise 10.1: some of the possible proofs require an interchange of the order of two integrals ("ombytning af integrationsrækkefølgen") or an interchange of a sum and an integral. In the concrete cases these interchanges can be justified by results in Section 5.3, but this is not required for this exercise.

Hint to Exercise 10.8: look directly at the expression of $\widehat{N_m}(2\gamma)$ and try to split it as a product of two terms, one of them being $\widehat{N_m}(\gamma)$

I am confident that this program is enough, but if you want an extra challenge you can look at the following exercise:

Problem K (thanks to Mathias Geisler!) Exercise 2.13 shows that if T is a bounded linear operator on a normed vector space V and W is a subspace of V, then

$$T(\overline{W}) \subseteq \overline{T(W)}. \tag{1}$$

It also shows that if T is invertible and T^{-1} is bounded, then we actually have that

$$T(\overline{W}) = \overline{T(W)}. \tag{2}$$

Intuitively, one could believe that the result in (2) always hold, but this is not true. In order to make a concrete example, it is enough to find an operator with non-closed range (think about this), and this is exactly what we will do now.

Let $\{e_k\}_{k=1}^{\infty}$ denote an orthonormal basis for a Hilbert space $\mathcal{H}$, and consider the vectors $v_k, k \in \mathbb{N}$, defined via

$$v_k := e_k + e_{k+1}.$$

In Problem 227 it is shown (you might do this) that $\overline{span}\{v_k\}_{k=1}^{\infty} = \mathcal{H}$, and that there does not exist complex numbers c_k such that $e_1 = \sum_{k=1}^{\infty} c_k v_k$.

Now, consider the operator T defined by

$$T : \ell^2(\mathbb{N}) \to \mathcal{H}, \ T\{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k v_k.$$

(i) Show that T is well defined, linear, and bounded.

(ii) Let $W := \ell^2(\mathbb{N})$ and show that (2) does NOT hold.

Homework 12, to be turned in no later than Wednesday May 15: 10.1 (only proof of Theorem 10.1.3 (ii)), 10.9

Regards,
Ole