01325 Mathematics 4, Spring 2013

Week no. 10

Theory: Wednesday April 17 the lectures cover Sections 7.2–7.4 and Sections 8.1–8.2 (until ca. page 164).

Remark: One of the exercises on the program (Exercise 8.1 (iii)) uses results that are not treated in the course. You might skip this subquestion if you want (but by doing the exercise you will realize how much theory we need in order to understand wavelets!)

Exercises for Wednesday April 17: 7.11, 8.1, 7.9, 2.8

Comments/hints to the exercises:

Ex. 8.1 (i): It will help you to make a small draft of a typical function in each of the spaces V_{-1}, V_0, and V_1.

Ex. 8.1 (iii): can also be solved using Theorem 8.2.11, but using the results stated in the exercise makes it more transparent *why* we obtain the result.

I am confident that this program is enough, but if you want an extra challenge you can look at Problem F below. The result implies that each of the spaces V_j in a multiresolution analysis is a Hilbert space.

Problem F Let $\mathcal{H}$ denote a Hilbert space. Show that any closed subspace V of $\mathcal{H}$ itself forms a Hilbert space.

Homework 10, to be turned in no later than Wednesday April 24: 5.19 (see the comment below) and Problem G below,

Comment to Exercise 5.19: I suggest that you split (i) into the following two steps:

(I) Analog to what we have seen for the Fourier transform and the blurring operator (Week 6) you first have to show that the expression $(Tf)(x)$ makes sense, i.e., that $\int_0^x |tf(t)|\,dt < \infty$ for all $f \in L^1(0,2)$ and all $x \in [0,2]$.

(II) In order to show that T maps $L^1(0,2)$ into $L^1(0,2)$ we need to show that if $f \in L^1(0,2)$, then

$$\int_0^2 |Tf(x)|\,dx < \infty.$$

In order to do so, show first that

$$|Tf(x)| \leq 2 \int_0^2 |f(t)|\, dt, \ \forall x \in [0, 2].$$

Problem G Let $p \in [1, \infty[$ and consider the mapping

$$T : L^p(-2, 2) \to L^p(-2, 2), \ (Tf)(x) := xf(x).$$

(i) Show that T indeed maps $L^p(-2, 2)$ into $L^p(-2, 2)$.

(ii) Show that T is linear and bounded.

Our aim is now to find the precise value of the number $||T||$.

(iii) Let $f = \chi_{[a,2]}$ for some $a \in]0, 2[$ and show that

$$a\, ||f||_p \leq ||Tf||_p \leq 2\, ||f||_p.$$

Hint: Do not calculate $||f||_p$ and $||Tf||_p$ explicitly, but look at the expression for $||Tf||_p$ for the given function $f = \chi_{[a,2]}$ and make some estimates, using that only the values of x belonging to the interval $[a, 2]$ are relevant.

(iv) Look at the inequalities you derived in (iii). Is it possible that, for example, $||T|| = 1.8$?

(v) Find the exact value of the number $||T||$.

Regards,
Ole