01325 Mathematics 4, Spring 2013

Week no. 6

Theory: In the week March 11–15 the lectures cover

Section 5.4 (Th. 5.4.2 will be discussed in more detail later) Sections 6.1, 6.2

Note that Section 6.4 gives an explanation of the difficulties with pointwise convergence of Fourier series that you experiences in Mathematics 2.

Exercises for the week March 11–15: 6.1, 6.4 (only for E_b. Remember to check that it is well defined, linear, bounded), 1.17, 5.9 (hint below), 6.11, Problem 104 below

Hint to Exercise 5.9 (i): consider the function $f(x) = \frac{1}{2\sqrt{x}}\chi_{[0,1]}(x)$.

Problem 104 Let

$$V := \{f \in L^2(\mathbb{R}) \mid f \text{ is infinitely often differentiable}, f' \in L^2(\mathbb{R}), [x \mapsto xf(x)] \in L^2(\mathbb{R})\}.$$

Consider the mappings defined by

$$D : V \to V, (Df)(x) := f'(x),$$

and

$$M : V \to V, (Mf)(x) := xf(x).$$

Note that the domain V is chosen such that the operators D and M are well defined.

(i) Show that D and M are linear mappings

(ii) Calculate the operator

$$DM - MD.$$

The operator $DM - MD$ is called the *commutator* of the operators D and M, and are used extensively in mathematical analysis and physics. It is usually denoted by $[D, M]$.

I am confident that this program is enough, but if you want an extra challenge you can look at Exercise 4.30.
Note: One can prove that if a bounded operator on a Hilbert space is bijective, then the inverse operator is automatically bounded. You might use this without proof in Exercise 4.30.

Homework 6, to be turned in no later than March 20: 6.4 (only for D_c), 6.3, Problem B (see below)

Problem B In image analysis, a *blur* can be used, e.g., to reduce the details in an image, or remove noise. Mathematically, a blur is described via an integral operator of the type discussed in Example 2.4.3, however, with the interval $[a, b]$ replaced by a 2-dimensional set because an image is 2-dimensional. We will restrict our attention to the 1-dimensional setting.

Frequently, a *Gaussian blur* is used. This means that the relation between the given signal ("image") x and the resulting blurred signal y is given by

$$y(t) = \int_a^b k(s,t)\, x(s)\, ds,$$

where

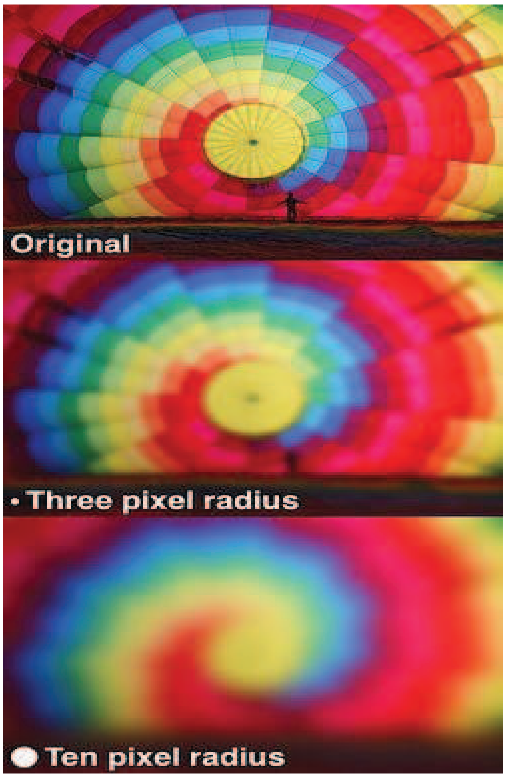
$$k(s,t) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(s-t)^2}{2\sigma^2}}$$

for a suitable chosen parameter $\sigma > 0$. The effect of the blurring is illustrated on the Figure, for, respectively, a small and a larger value of σ.

Motivated by the blurring example, we want to consider a linear operator K, that maps a given signal x onto a new signal y, given by

$$y(t) = (Kx)(t) := \int_{-\infty}^{\infty} e^{-(s-t)^2}\, x(s)\, ds,\ \ t \in \mathbb{R}.$$

We would like to make sure that the integral defining the function y actually is well defined for all $t \in \mathbb{R}$. The purpose of this exercise is to check whether this is the case for various vector spaces. Note that in order to show that an operator is *not* well defined on a certain vector space, it is enough to find just *one* function in the vector space, for which it is not well defined.



(i) Is $y(t) := (Kx)(t)$ well defined for all $x \in C(\mathbb{R})$ and all $t \in \mathbb{R}$?

(ii) Is $y(t) := (Kx)(t)$ well defined for all $x \in C_0(\mathbb{R})$ and all $t \in \mathbb{R}$?
Hint: you might use that

$$\int_{-\infty}^{\infty} e^{-u} \, du = \sqrt{\pi}.$$

(iii) Is $y(t) := (Kx)(t)$ well defined for all $x \in L^1(\mathbb{R})$ and all $t \in \mathbb{R}$?

More information about image analysis and blurring is given in the course 02625, which uses the book *Deblurring images* by Per Christian Hansen, J.G. Nagy, and D.P O'Leary.

Regards,
Ole