Extra exercises, 01325 Mathematics 4

Problem 1: Let V denote the subset of $C[-\pi, \pi]$ consisting of all finite linear combinations of functions

$$1, \cos x, \cos 2x, \ldots, \cos nx, \ldots, \sin x, \sin 2x, \ldots \sin nx, \ldots.$$

(i) Show that V is a subspace of $C[-\pi, \pi]$.

(ii) Is V closed in $C[-\pi, \pi]$? Hint: from Fourier analysis (e.g., example 6.17 in the MAT 2 book from 2009) it is known that for $x \in [-\pi, \pi]$,

$$\left| x^2 - \left(\frac{\pi^2}{3} + 4 \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos nx \right) \right| \leq 4 \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

You may use this result without proof.

Problem 4: Consider the following functions defined on $\mathbb{R}$ (see p.20 for the definition of $\chi_{[a,b]}$):

$$\begin{aligned} f_1(x) &= x, \\ f_2(x) &= x^2 - 1, \\ f_3(x) &= (x^2 - 1)\chi_{[-1,2]}(x), \\ f_4(x) &= (x^2 - 1)\chi_{[-1,1]}(x), \\ f_5(x) &= e^{-|x|}, \end{aligned}$$

(i) Make a rough sketch of the graph of each of the functions.

(ii) Which functions have compact support? Determine the support for these functions.

(iii) Which functions belong to $C_0(\mathbb{R})$?

(iv) Which functions belong to $C_c(\mathbb{R})$?

(v) Which functions belong to $L^1(\mathbb{R})$?

Problem 6:

(i) Show that we can define an equivalence relation on $L^1(\mathbb{R})$ by

$$f \sim g \Leftrightarrow \int_{-\infty}^{\infty} |f(x) - g(x)|\, dx = 0.$$

Now let $\sim$ denote an equivalence relation on any set V.

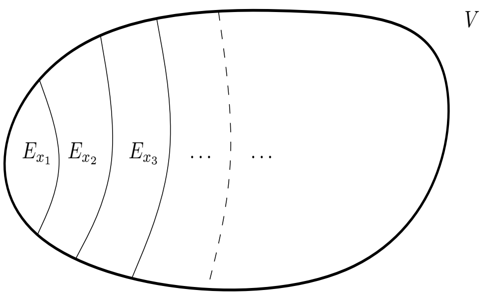
(ii) Show that for any $x_1, x_2 \in V$ we have

$$E_{x_1} = E_{x_2} \;\text{ or }\; E_{x_1} \cap E_{x_2} = \emptyset.$$

(iii) Argue that there exists a collection $\{E_{x_i}\}_{i \in I}$ of elements in V such that

$$V = \bigcup_{i \in I} E_{x_i} \;\text{ and }\; E_{x_i} \cap E_{x_j} = \emptyset \text{ if } i \neq j.$$

In words, the result in (iii) shows that an equivalence relation splits the set V into a disjoint union of equivalence classes. In the case of $L^1(\mathbb{R})$ we obtain a normed space if we identify all elements within a given equivalence class. Thus, strictly speaking $L^1(\mathbb{R})$ is not a space of functions, but a space of equivalence classes of functions.



Problem 7 Let $p \in [1, \infty[$ and consider the mapping

$$T : L^p(-2,2) \to L^p(-2,2),\; (Tf)(x) := xf(x).$$

(i) Show that T indeed maps $L^p(-2,2)$ into $L^p(-2,2)$.

(ii) Show that T is linear and bounded.

(iii) Calculate the norm of the operator T.

Hint: let $f = \chi_{[\epsilon,2]}$ for some $\epsilon > 0$ and show that

$$\epsilon\, ||f||_p \leq ||Tf||_p \leq 2\, ||f||_p$$

Problem 8 Consider the function

$$f(x) = \frac{1}{|x|^{1/3}}\,\chi_{]0,\infty[}(x),\ \ x \in \mathbb{R}.$$

(i) Make a draft of the function f.

(ii) Check whether there exists any $p > 0$ for which $f \in L^p(\mathbb{R})$.

Problem 9 Consider the vector space $\mathbb{C}^2$. For $\mathbf{x} \in \mathbb{C}^2$, we write $\mathbf{x} = (x_1, x_2),\ x_1, x_2 \in \mathbb{C}$.

(i) Does the expression

$$||(x_1, x_2)|| := |x_1|$$

define a norm on $\mathbb{C}^2$?

(ii) Does the expression

$$||(x_1, x_2)|| := |x_1| - |x_2|$$

define a norm on $\mathbb{C}^2$?

(iii) Does the expression

$$||(x_1, x_2)|| := \max\left(|x_1|, |x_2|\right)$$

define a norm on $\mathbb{C}^2$?

Problem 101 (extension of Problem 221 from the Exam 2011) Consider the function

$$f(x) := e^x \chi_{[0,1]}(x).$$

(i) Calculate the convolution $(f * f)(x)$, $x \in \mathbb{R}$.

(ii) Calculate the Fourier transform

$$\widehat{f * f}(\gamma),\ \ \gamma \in \mathbb{R}.$$

Problem 102 Let $r : [a,b] \to]1,\infty[$ be a continuous function and consider the weighted L^2-space

$$L_r^2(a,b) = \{f : [a,b] \to \mathbb{C} \mid \int_a^b |f(x)|^2 r(x)\, dx < \infty\}.$$

We know (you are not expected to show this) that $L_r^2(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_r = \int_a^b f(x)\overline{g(x)} r(x)\, dx.$$

(i) Show that $L_r^2(a,b)$ is a subspace of $L^2(a,b)$.

(ii) Show that if $\{e_k(x)\}_{k=1}^{\infty}$ is an orthonormal system in $L_r^2(a,b)$, then $\{e_k(x)\sqrt{r(x)}\}_{k=1}^{\infty}$ is an orthonormal system in $L^2(a,b)$.

Problem 103 Consider the Legendre differential equation, which, for a given and fixed parameter $\lambda \in \mathbb{R}$, is

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + \lambda u = 0. \tag{1}$$

Assume that (1) has a power series solution,

$$u(x) = \sum_{k=0}^{\infty} c_k x^k.$$

(i) Show that for $k \geq 2$,

$$c_{k+2}(k+2)(k+1) - c_k(k(k+1) - \lambda).$$

(ii) Show that if (1) has a solution $u \neq 0$ which is a polynomial, then necessarily the parameter λ has the form

$$\lambda = \ell(\ell+1)$$

for some $\ell \in \{0, 1, 2, \dots\}$.

Problem 104 Let

$$V := \{f \in L^2(\mathbb{R}) \mid f \text{ is infinitely often differentiable}, f' \in L^2(\mathbb{R}), [x \mapsto xf(x)] \in L^2(\mathbb{R})\}.$$

Consider the mappings defined by

$$D : V \to V, (Df)(x) := f'(x),$$

and

$$M : V \to V, (Mf)(x) := xf(x).$$

The domain V is chosen such that the operators D and M are well defined.

(i) Show that D and M are linear mappings

(ii) Calculate the operator

$$DM - MD.$$

Problem 105 We will consider the modulation operator E_b for complex values of the parameter b. That is, for $b \in \mathbb{C}$ we let E_b act on the function $f : \mathbb{R} \to \mathbb{C}$ by

$$(E_b f)(x) := e^{2\pi ibx} f(x), \ x \in \mathbb{R}.$$

(i) Show that the function

$$f(x) := \frac{1}{\sqrt{1+x^2}}$$

belongs to $L^2(\mathbb{R})$.

(ii) For which values of the parameter $b \in \mathbb{C}$ does the operator E_b define a bounded operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$?

Hint: Find the values of $b \in \mathbb{C}$ for which $E_b f \in L^2(\mathbb{R})$ for the function f in (i).

Problem 106 (Extension of Problem 222, Exam 2011) Consider a complex Hilbert space $\mathcal{H}$ and a bounded linear operator

$$T : \mathcal{H} \to \mathcal{H}.$$

We say that a number $\lambda \in \mathbb{C}$ is an eigenvalue for T if there exists an eigenvector, i.e., a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$T\mathbf{v} = \lambda\,\mathbf{v}.$$

(i) Assume that $\lambda \in \mathbb{C}$ is an eigenvalue for T, and let $\mathbf{v}$ denote a corresponding eigenvector with $||\mathbf{v}|| = 1$. Calculate the numbers

$$\langle T\mathbf{v}, \mathbf{v}\rangle \quad \text{and} \quad \langle \mathbf{v}, T\mathbf{v}\rangle.$$

(ii) Assume that T is self-adjoint. Show that all the eigenvalues are real numbers.

Now assume that $\mathcal{H}$ has an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and that for each $k \in \mathbb{N}$ there exists a number $\lambda_k \in \mathbb{R}$ such that

$$Te_k = \lambda_k e_k.$$

(iii) Show that for any $f \in \mathcal{H}$,

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k\rangle\, e_k.$$

Problem 107 Assume that a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx}, \tag{2}$$

where the series $\sum_{n=-\infty}^{\infty} d_n$ is absolutely convergent. The goal of this exercise is to show that then the coefficients d_n are necessarily the well known Fourier coefficients on complex form.

(i) Show that the series in (2) has a convergent majorant series.

(ii) Calculate for each $m, n \in \mathbb{Z}$ the integral

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx.$$

Hint: consider the cases $m = n$ and $m \neq n$ separately.

(iii) Prove using (ii) that for any $m \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} d_n e^{inx}\, e^{-imx}\, dx = 2\pi d_m.$$

Hint: The order of integration and summation can be interchanged by Theorem 5.34 in the MAT 2 book from 2012 (Theorem 5.33 in the version from 2009), or see Theorem 5.3.7 in the MAT4 book.

(iv) Conclude using (iii) that

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx, \;\; \forall n \in \mathbb{Z}.$$

Problem 108 (*This is a quite unusual exercise.*)

Consider the function

$$f(x) := \frac{1}{1+x^2}, \;\; x \in \mathbb{R}.$$

(i) Show that $f \in L^2(\mathbb{R})$.

Hint: By Exercise 5.3 we know that $f \in L^1(\mathbb{R})$.

(ii) Does the function f have any vanishing moments?

(iii) Argue that there exists a function $\phi \in L^2(\mathbb{R})$ such that

$$\widehat{\phi}(\gamma) = \frac{1}{1+\gamma^2}, \gamma \in \mathbb{R}. \tag{3}$$

Hint: Use a result in Section 7.2

(iv) One can show that the function ϕ in (iii) belongs to $L^1(\mathbb{R})$. Does the function ϕ have any vanishing moments?

(v) Does the function ϕ in (iii) generates a multiresolution analysis?

Technical University of Denmark

Written exam. Date: June 6, 2008

Course name: Mathematics 4: Real Analysis, Course no. 01325

Part I : Problems 201 – 204 (6 questions)
Part II: Problems 5 – 7 (6 questions)

The exam consists of 7 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 201 (contained in Ex. 3.9) Consider a sequence $\{w_k\}_{k=1}^{\infty}$ of positive real numbers, and define the weighted ℓ^1-space $\ell^1_w(\mathbb{N})$ by

$$\ell^1_w(\mathbb{N}) := \left\{ \{x_k\}_{k=1}^{\infty} \mid x_k \in \mathbb{C}, \ \sum_{k=1}^{\infty} |x_k|\, w_k < \infty \right\}.$$

(i) Show that the expression $||\cdot||$ given by

$$||\{x_k\}_{k=1}^{\infty}|| := \sum_{k=1}^{\infty} |x_k|\, w_k$$

defines a norm on $\ell^1_w(\mathbb{N})$.

We now consider the special choice

$$w_k := 2^k, \ k \in \mathbb{N}.$$

(ii) Show that $\ell^1_w(\mathbb{N})$ is a subspace of $\ell^1(\mathbb{N})$.

The set of problems continues!

Problem 202 (Contained in Ex. 6.10) Consider the linear operator

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \ (Uf)(x) = f(2x-2), \ x \in \mathbb{R}.$$

(i) Show that U is bounded on $L^2(\mathbb{R})$.

(ii) Compute the adjoint operator U^*.

Problem 203 (Ex. 10.6) Consider the symmetric B-splines B_2 and B_3. Find an expression for the Fourier transform of the convolution $B_2 * B_3$, i.e., determine the function

$$\widehat{B_2 * B_3}.$$

Problem 204 (Ex. 4.2) Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $|| \cdot ||$. Show that for all $\mathbf{u}, \mathbf{v} \in \mathcal{H}$,

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2\left(||\mathbf{u}||^2 + ||\mathbf{v}||^2\right).$$

Problem 5-7 Taken out, as they are not relevant for Mathematics 4 in its present version.

The set of problems is completed!

Technical University of Denmark

TEST!! Date: May 6, 2009

Course name: Mathematics 4, Course no. 01325

Allowed aids: All aids.

The test consists of 6 problems, all together with 12 questions. The 12 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 205 (Ex. 1.7) Consider the function

$$f(x) := -xe^{-x},\ x \in [0, \infty[.$$

Calculate the number

$$\inf\{f(x) \mid x \in [0, \infty[\}.$$

Can "infimum" be replaced by "minimum"?

Problem 206 (Ex. 6.15) For $f \in L^2(-\pi, \pi)$, the complex Fourier coefficients are defined by

$$c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx,\ k \in \mathbb{Z}.$$

Show that the integral defining c_k is well defined for $f \in L^2(-\pi, \pi)$, i.e., that

$$\int_{-\pi}^{\pi} \left| f(x) e^{-ikx} \right| dx < \infty.$$

The set of problems continues!

Problem 207 (Ex. 4.15) Let $\{\mathbf{v}_k\}_{k=1}^{\infty}$ denote a sequence in a Hilbert space $\mathcal{H}$. The purpose is to prove Lemma 4.4.4. in the book, which claims that (a) and (b) below are equivalent:

(a) $\{\mathbf{v}_k\}_{k=1}^{\infty}$ is complete;

(b) If $\mathbf{v} \in \mathcal{H}$ and $\langle \mathbf{v}, \mathbf{v}_k \rangle = 0$ for all $k \in \mathbb{N}$, then $\mathbf{v} = \mathbf{0}$.

Do the following:

(i) Show that if $\mathbf{v} \in \mathcal{H}$ and $\langle \mathbf{v}, \mathbf{v}_k \rangle = 0$ for all $k \in \mathbb{N}$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty}$.
Hint: show first that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in \text{span}\{\mathbf{v}_k\}_{k=1}^{\infty}$.

(ii) Use (i) to show that if (a) holds, then (b) holds.

(iii) Prove that if (a) does not hold, then (b) does not hold.
Hint: Let $W := \overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty}$, and use Theorem 4.3.5 to argue that if (a) does not hold, then $W^{\perp} \neq \{\mathbf{0}\}$.

Problem 208 (Ex. 5.17) Let $p \in [1, \infty[$ and consider the mapping

$$T : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \ (Tf)(x) := f(3x + 2).$$

(i) Show that T indeed maps $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$.

(ii) Show that T is linear and bounded.

(iii) Consider the function

$$f(x) := \frac{1}{x^2} \chi_{[1,\infty[}(x),$$

and show that $f \in L^p(\mathbb{R})$ for all $p \in [1, \infty[$.

The set of problems continues!

Problem 209 Let ψ denote the Haar wavelet. We know that ψ can be constructed from the multiresolution analysis generated by the function $\phi = \chi_{[0,1[}$.

(i) Show that for $f \in L^2(\mathbb{R})$,

$$\langle f, T_k\phi \rangle = \int_0^1 f(x+k)\, dx.$$

(ii) Consider a compactly supported function $f \in L^2(\mathbb{R})$ and the expansion (8.14) in the book. Show that the term

$$\sum_{k\in\mathbb{Z}} \langle f, T_k\phi \rangle T_k\phi$$

actually is a finite sum. *Hint:* Assume that $\operatorname{supp} f \subseteq [-N, N]$ and consider $c_k := \langle f, T_k\phi \rangle$ for $|k| \geq N+1$.

Problem 210 (Ex. 8.11) Let $\phi \in L^2(\mathbb{R})$, and denote its Fourier transform by $\hat{\phi}$.

(i) Show that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system if and only if

$$\langle \phi, T_k\phi \rangle = \begin{cases} 1 & \text{if } k = 0; \\ 0 & \text{if } k \neq 0. \end{cases}$$

Let

$$\Phi(\gamma) := \sum_{n\in\mathbb{Z}} |\hat{\phi}(\gamma+n)|^2.$$

One can show **(you are not expected to do this)** that

$$\langle \phi, T_k\phi \rangle = \int_0^1 \Phi(\gamma) e^{2\pi i k\gamma}\, d\gamma.$$

(ii) Show that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal system if and only if $\Phi(\gamma) = 1$ for $\gamma \in \mathbb{R}$.

Hint: By Example 6.4.3 in the book, the numbers $\int_0^1 \Phi(\gamma) e^{2\pi i k\gamma}\, d\gamma$ are Fourier coefficients for the 1-periodic function Φ.

The set of problems is completed!

Technical University of Denmark

Written exam. Date: June 4, 2009

Course name: Mathematics 4, Course no. 01325

Allowed aids: All aids.

The exam consists of 5 problems, all together with 10 questions. The 10 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 211 Consider the function

$$f(x) = x - x^2.$$

Determine the number

$$\sup\{f(x) \big| \; x \in [0,4]\}.$$

Problem 212 (Ex. 4.6) Consider the vector space $\ell^p(\mathbb{N})$ for some $p \in [1, \infty[$, equipped with the usual norm

$$||\{x_k\}_{k=1}^{\infty}||_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}.$$

(i) Consider the vectors

$$\mathbf{x} = (1, 0, 0, \dots), \quad \mathbf{y} = (0, 1, 0, 0, \dots),$$

and show that

$$||\mathbf{x}||_p = ||\mathbf{y}||_p = 1, \quad ||\mathbf{x} + \mathbf{y}||_p = ||\mathbf{x} - \mathbf{y}||_p = 2^{1/p}.$$

(ii) Assume that $p \neq 2$. Show that the norm $|| \cdot ||_p$ does not come from an inner product. *Hint: use Theorem 4.1.4.*

Problem 213 Consider a Hilbert space $\mathcal{H}$ and let W be a closed subspace of $\mathcal{H}$. Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ be an orthonormal basis for W. Define the operator $P : \mathcal{H} \to \mathcal{H}$ by

$$P\mathbf{v} := \sum_{k=1}^{\infty} \langle \mathbf{v}, \mathbf{e}_k \rangle \mathbf{e}_k, \ \ \mathbf{v} \in \mathcal{H}.$$

(i) Show that if $\mathbf{w} \in W$ and $\mathbf{u} \in W^{\perp}$, then

$$P(\mathbf{w} + \mathbf{u}) = \mathbf{w}.$$

(ii) Show that if $\mathbf{w} \in W$ and $\mathbf{u} \in W^{\perp}$, then

$$||\mathbf{w} + \mathbf{u}||^2 = ||\mathbf{w}||^2 + ||\mathbf{u}||^2.$$

(iii) Show that $||P\mathbf{v}|| \leq ||\mathbf{v}||$ for all $\mathbf{v} \in \mathcal{H}$.
Hint: By Theorem 4.3.5, any $\mathbf{v} \in \mathcal{H}$ can be written

$$\mathbf{v} = \mathbf{w} + \mathbf{u} \text{ for some } \mathbf{w} \in W, \ \ \mathbf{u} \in W^{\perp}.$$

Problem 214 (Ex. 5.20) Let $p \in [1, \infty[$ and consider the linear mapping

$$T : L^p(0,2) \to L^p(0,2), \ \ (Tf)(x) := xf(x).$$

One can show **(you are not expected to do this)** that for $0 \leq a \leq b$,

$$a^p \int_a^b |f(x)|^p \, dx \leq \int_a^b |xf(x)|^p \, dx \leq b^p \int_a^b |f(x)|^p \, dx. \tag{4}$$

(i) Use (4) to argue that T indeed maps $L^p(0,2)$ into $L^p(0,2)$ and is bounded with

$$||T|| \leq 2.$$

(ii) Calculate the norm of the operator T.
Hint: let $f = \chi_{[\epsilon,2]}$ for some $\epsilon \in [0, 2[$ and use (4) to show that

$$\epsilon \leq \frac{||Tf||_p}{||f||_p} \leq 2.$$

Problem 215 (Contained in Ex. 10.11) Let $m \in \mathbb{N}$, and consider the B-spline N_m. One can show **(you are not expected to do this)** that there exist constants $A, B > 0$ such that

$$A \leq \sum_{k \in \mathbb{Z}} |\widehat{N_m}(\gamma + k)|^2 \leq B, \ \gamma \in \mathbb{R}.$$

(i) Show that the function

$$G(\gamma) := \sum_{k \in \mathbb{Z}} |\widehat{N_m}(\gamma + k)|^2$$

is 1-periodic.

Define the function $\varphi \in L^2(\mathbb{R})$ by its Fourier transform $\widehat{\varphi}$ via

$$\widehat{\varphi}(\gamma) := \frac{1}{\sqrt{G(\gamma)}} \widehat{N_m}(\gamma).$$

(ii) Show that

$$\sum_{k \in \mathbb{Z}} |\widehat{\varphi}(\gamma + k)|^2 = 1, \ \gamma \in \mathbb{R}.$$

The set of problems is completed!

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Written exam. Date: June 1, 2010.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

The exam consists of 4 problems, all together with 10 questions. The 10 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 216 Consider the set of functions

$$V := \{f : \mathbb{R} \to \mathbb{C} \mid f(-\pi) = f(\pi) = 0\}.$$

(i) Show that V is a vector space with respect to the usual operations of addition and scalar multiplication.

(ii) Give an example of a linear operator that does not map V into V.

The set of problems continues!

Problem 217 Consider the linear mapping

$$T : L^1(0,2) \to L^1(0,2), \ (Tf)(x) := \int_0^2 e^{-x^2y^2} f(y)\, dy.$$

(i) Show that for any $x \in [0,2]$,

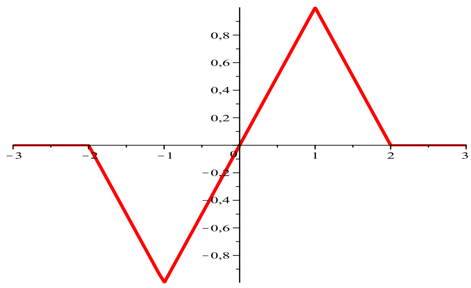
$$|(Tf)(x)| \le ||f||_{L^1(0,2)}.$$

(ii) Use the result in (i) to show that T indeed maps $L^1(0,2)$ into $L^1(0,2)$ and is bounded with

$$||T|| \le 2.$$

Problem 218 Consider the B-spline N_2 and its translate $T_{-2}N_2$. Define the function f by

$$f(x) = N_2(x) - T_{-2}N_2(x).$$



The graph shows the function f. The following questions can be answered without calculating the function f explicitly.

(i) Show that the Fourier transform of the function f is

$$\hat{f}(\gamma) = \left(1 - e^{4\pi i\gamma}\right)\left(\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right)^2.$$

(ii) Find the number of vanishing moments for the function f.
Hint: Instead of calculating the involved integrals you can use symmetry properties of the functions $f(x)$ and $xf(x)$.

Problem 219 Consider a Hilbert space $\mathcal{H}$ and let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Consider a bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$ for which the inverse T^{-1} is bounded as well.

(i) Show that if $\mathbf{v} \in \mathcal{H}$, then

$$\sum_{k=1}^{\infty} |\langle \mathbf{v}, T\mathbf{e}_k \rangle|^2 = ||T^*\mathbf{v}||^2.$$

(ii) Show that the adjoint operator T^* is invertible, and that

$$(T^*)^{-1} = (T^{-1})^*.$$

Hint: Use the result in Exercise 4.17 (ii).

(iii) Show that for all $\mathbf{v} \in \mathcal{H}$

$$||\mathbf{v}|| \leq ||(T^*)^{-1}|| \, ||T^*\mathbf{v}||.$$

Hint: Use that $\mathbf{v} = (T^*)^{-1}T^*\mathbf{v}$.

(iv) Show that for all $\mathbf{v} \in \mathcal{H}$

$$\frac{1}{||T^{-1}||^2} \, ||\mathbf{v}||^2 \leq \sum_{k=1}^{\infty} |\langle \mathbf{v}, T\mathbf{e}_k \rangle|^2 \leq ||T||^2 \, ||\mathbf{v}||^2.$$

The set of problems is completed!

Technical University of Denmark

Written exam. Date: May 31, 2011.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

Duration: 2 Hours.

The exam consists of 4 problems, all together with 10 questions. The 10 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

Problem 220 Consider the function

$$f(x) := \sum_{k=2}^{\infty} k\,\chi_{[k,k+k^{-3}]}(x) = 2\,\chi_{[2,2\frac{1}{8}]}(x) + 3\,\chi_{[3,3\frac{1}{27}]}(x) + \cdots,\ x \in \mathbb{R}.$$

(i) Make a draft of the function f.

(ii) Is f a bounded function?

(iii) Does $f \in L^1(\mathbb{R})$?

Problem 221 Consider the function

$$f(x) := e^x \chi_{[0,1]}(x),\ x \in \mathbb{R}.$$

(i) Calculate the convolution $(f * f)(x)$ for $x \in [0,1]$.

The set of problems continues!

Problem 222 Consider a complex Hilbert space $\mathcal{H}$ and a bounded, linear, and self-adjoint operator

$$T : \mathcal{H} \to \mathcal{H}.$$

We say that a number $\lambda \in \mathbb{C}$ is an eigenvalue for T if there exists an eigenvector, i.e., a vector $\mathbf{v} \neq \mathbf{0}$ such that

$$T\mathbf{v} = \lambda\,\mathbf{v}.$$

(i) Assume that $\lambda \in \mathbb{C}$ is an eigenvalue for T, and let $\mathbf{v}$ denote a corresponding eigenvector with $||\mathbf{v}|| = 1$. Calculate the numbers

$$\langle T\mathbf{v}, \mathbf{v}\rangle \text{ and } \langle \mathbf{v}, T\mathbf{v}\rangle.$$

(ii) Show that all the eigenvalues are real numbers.

Now assume that $\mathcal{H}$ has an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and that for each $k \in \mathbb{N}$ there exists a number $\lambda_k \in \mathbb{R}$ such that

$$Te_k = \lambda_k e_k.$$

(iii) Show that for any $f \in \mathcal{H}$,

$$Tf = \sum_{k=1}^{\infty} \lambda_k \langle f, e_k\rangle\, e_k.$$

Problem 223 We will consider the modulation operator E_b for complex values of the parameter b. That is, for $b \in \mathbb{C}$ we let E_b act on the function $f : \mathbb{R} \to \mathbb{C}$ by

$$(E_b f)(x) := e^{2\pi i b x} f(x), \ x \in \mathbb{R}.$$

(i) Show that the mapping E_b is linear for any $b \in \mathbb{C}$.

(ii) Show that for any $b \in \mathbb{C}$, the operator E_b maps $L^2(0,1)$ into $L^2(0,1)$ and is bounded.

Hint: Write $b = \alpha + i\beta$ *for some* $\alpha, \beta \in \mathbb{R}$, *and use that every exponential function is bounded on* $[0,1]$.

(iii) Determine the values of the parameter $b \in \mathbb{C}$ for which the operator $E_b : L^2(0,1) \to L^2(0,1)$ is self-adjoint.

The set of problems is completed!

Technical University of Denmark

Written exam. Date: May 29, 2012.

Course name: Mathematics 4, Course no. 01325.

Allowed aids: All aids.

Duration: 2 Hours.

The exam consists of 4 problems, all together with 11 questions. The 11 questions are weighted equally. In order to obtain full credit, you are required to provide complete arguments. The answers will be judged as a whole. The answers can be given in English or Danish.

All references are to the book "Functions, Spaces, and Expansions, 2010" by Ole Christensen.

Problem 224 Let $\omega : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function, and consider the mapping

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\ (Uf)(x) = \omega(x)f(x),\ x \in \mathbb{R}.$$

(i) Show that U indeed maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.

(ii) Show that U is linear and bounded.

(iii) Compute the adjoint operator U^*.

(iv) Show that U is unitary if and only if

$$|\omega(x)| = 1,\ \forall\, x \in \mathbb{R}.$$

The set of problems continues!

Problem 225 We consider the space $L^p(0,\pi/2)$ for $p = 1/2$, i.e.,

$$L^{1/2}(0,\pi/2) := \{f :]0,\pi/2[\to \mathbb{C} \mid \int_0^{\pi/2} |f(x)|^{1/2}\, dx < \infty\}.$$

(i) Does the function

$$h(x) := \frac{1}{x},\ \ x \in]0,\pi/2[$$

belong to $L^{1/2}(0,\pi/2)$?

(ii) Show that every bounded function $f :]0,\pi/2[\to \mathbb{C}$ belongs to $L^{1/2}(0,\pi/2)$.

One can show (**you are not expected to do this**) that the functions

$$f(x) := \sin^2(x), \quad g(x) := \cos^2(x), \quad x \in]0,\pi/2[,$$

satisfy that

$$\begin{aligned} \int_0^{\pi/2} |f(x)|^{1/2}\, dx &= 1, \\ \int_0^{\pi/2} |g(x)|^{1/2}\, dx &= 1, \\ \int_0^{\pi/2} |f(x) + g(x)|^{1/2}\, dx &= \frac{\pi}{2}. \end{aligned}$$

(iii) Does the expression

$$||f|| := \left(\int_0^{\pi/2} |f(x)|^{1/2}\, dx \right)^2$$

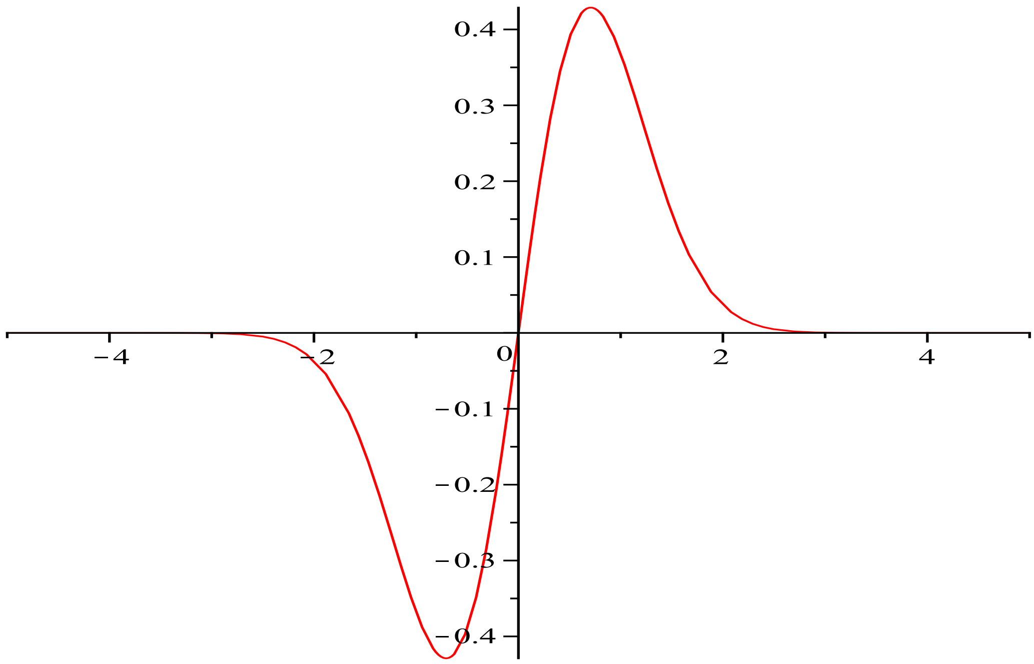
define a norm on $L^p(0,\pi/2)$ for $p = 1/2$?

The set of problems continues!

Problem 226 The figure shows the graph of the function

$$f(x) := xe^{-x^2}, \ x \in \mathbb{R}.$$

Find the number of vanishing moments for the function f. (An argument based on the properties of the function f is sufficient in order to obtain full credit.)



The set of problems continues!

Problem 227 Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ denote an orthonormal basis for a Hilbert space $\mathcal{H}$, and let

$$\mathbf{v}_k := \mathbf{e}_k + \mathbf{e}_{k+1}, \ k \in \mathbb{N}.$$

(i) Let $\mathbf{v} \in \mathcal{H}$ and assume that $\langle \mathbf{v}, \ \mathbf{v}_k \rangle = 0$ for all $k \in \mathbb{N}$. Show that then

$$|\langle \mathbf{v}, \mathbf{e}_k \rangle| = |\langle \mathbf{v}, \mathbf{e}_{k+1} \rangle|, \ \forall k \in \mathbb{N}.$$

(ii) Show that $\overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty} = \mathcal{H}$.

Hint: Use (i) and Theorem 4.7.2 (iv) to show that if $\langle \mathbf{v}, \mathbf{v}_k \rangle = 0$, $k \in \mathbb{N}$, then $\mathbf{v} = \mathbf{0}$.

One can show (**you are not expected to do this**) that for any $N \in \mathbb{N}$ and complex numbers $c_k \in \mathbb{C}$ for $k = 1, \ldots, N$,

$$\left\| \mathbf{e}_1 - \sum_{k=1}^{N} c_k \mathbf{v}_k \right\|^2 = |1 - c_1|^2 + \sum_{k=1}^{N-1} |c_k + c_{k+1}|^2 + |c_N|^2. \tag{5}$$

(iii) Show that it is impossible to choose $c_k \in \mathbb{C}$ such that

$$\mathbf{e}_1 = \sum_{k=1}^{\infty} c_k \mathbf{v}_k.$$

Hint: Use (5) to argue that we only need to consider the special sequence $c_k = (-1)^{k-1}, \ k \in \mathbb{N}$.

The set of problems is completed!