12.16 Week 13

12.16.1 Selected solutions

Exercise 11.4: For $\ell = 0$, we know that $P_0(x) = 1,\ x \in [-1, 1]$, so

$$\int_{-1}^{1} |P_0(x)|^2\, dx = 2,$$

as claimed by Lemma 11.2.6(ii). Now, assume that the result in the Lemma holds for some ℓ. Then, via (10.17) with ℓ replaced by $\ell + 1$,

$$\begin{aligned}\int_{-1}^{1} |P_{\ell+1}(x)|^2\, dx = \frac{2(\ell+1)-1}{2(\ell+1)+1} \int_{-1}^{1} |P_\ell(x)|^2\, dx &= \frac{2\ell+1}{2\ell+3}\frac{2}{2\ell+1} \\ &= \frac{2}{2(\ell+1)+1}.\end{aligned}$$

This shows that Lemma 11.2.6(ii) holds for all ℓ.

Problem 108: (i) Observe that $0 \leq \frac{1}{1+x^2} \leq 1$ for all $x \in \mathbb{R}$. Therefore

$$\left|\frac{1}{1+x^2}\right|^2 = \left(\frac{1}{1+x^2}\right)^2 \leq \frac{1}{1+x^2},\ \forall x \in \mathbb{R}.$$

Since $f \in L^2(\mathbb{R})$, it follows that

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} \left|\frac{1}{1+x^2}\right|^2 \leq \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx < \infty,$$

i.e., $f \in L^2(\mathbb{R})$.

(ii) Since $f(x) > 0$ for all $x \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} f(x)\, dx > 0.$$

Thus the function f does not have any vanishing moments.

(iii) By Theorem 7.2.2 the Fourier transform is a unitary operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. A unitary operator is always surjective. Since we have shown in (i) that $f \in L^2(\mathbb{R})$, we therefore know that there exists a function $\phi \in L^2(\mathbb{R})$ such that $\mathcal{F}\phi = f$, i.e.,

$$\widehat{\psi}(\gamma) = \frac{1}{1+\gamma^2}.$$

(iv) By the definition of the Fourier transform, we know that

$$\int_{-\infty}^{\infty} \phi(x)\, dx = \widehat{\phi}(0) = 1 \neq 0.$$

Thus, the function ϕ does not have any vanishing moments.

(v) If a function $\phi \in L^2(\mathbb{R})$ generates a multiresolution analysis, we know from Proposition 8.2.6 that there exists a 1-periodic function H_0 such that

$$\hat{\phi}(2\gamma) = H_0(\gamma)\hat{\phi}(\gamma), \ \gamma \in \mathbb{R}.$$

We know the expression for $\widehat{\phi}$, so the unique choice for H_0 is the function

$$H_0(\gamma) = \frac{\widehat{\phi}(2\gamma)}{\widehat{\phi}(\gamma)} = \frac{\frac{1}{1+(2\gamma)^2}}{\frac{1}{1+\gamma^2}} = \frac{1+\gamma^2}{1+4\gamma^2}.$$

This function is *not* 1-periodic. Thus, the function ϕ does not generate a multiresolution analysis.

Note that Problem 108 is a quite unusual exercise. It illustrates that if we just pick a random function, it is very unlikely that it generates a multiresolution analysis - and it is very unlikely that it has a high number of vanishing moments. Thus, the wavelet constructions are really very special!

Problem 101: (i) By definition

$$\begin{aligned}(f*f)(x) = \int_{-\infty}^{\infty} f(x-y)f(y)\,dy &= \int_{-\infty}^{\infty} e^{x-y}\chi_{[0,1]}(x-y)e^{y}\chi_{[0,1]}(y)\,dy \\ &= \int_{-\infty}^{\infty} e^{x}\chi_{[0,1]}(x-y)\chi_{[0,1]}(y)\,dy \\ &= e^{x}\int_{-\infty}^{\infty} \chi_{[0,1]}(x-y)\chi_{[0,1]}(y)\,dy \\ &= e^{x}(\chi_{[0,1]}*\chi_{[0,1]})(x) \\ &= e^{x}N_2(x),\end{aligned}$$

where N_2 is the B-spline of order 2. Using the formula for N_2 in Exercise 10.2, we see that

$$(f*f)(x) = \begin{cases} xe^x, & \text{if } x \in [0,1], \\ (2-x)e^x, & \text{if } x \in [1,2], \\ 0, & \text{if } x \notin [1,2]. \end{cases}$$

(ii) By direct calculation

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\gamma x}\,dx = \int_{0}^{1} e^{x(1-2\pi i\gamma}\,dx = \frac{e^{1-2\pi i\gamma}-1}{1-2\pi i\gamma}.$$

Thus,

$$\widehat{f*f}(\gamma) = \left(\hat{f}(\gamma)\right)^2 = \left(\frac{e^{1-2\pi i\gamma}-1}{1-2\pi i\gamma}\right)^2$$

Problem 217:(i) For $x \in [0,2]$,

$$
\begin{aligned}
|(Tf)(x)| &= \left|\int_0^2 e^{-x^2y^2} f(y)\,dy\right| \leq \int_0^2 \left|e^{-x^2y^2} f(y)\right|\,dy \\
&\leq \int_0^2 |f(y)|\,dy = ||f||_{L^1(0,2)}.
\end{aligned}
$$

The calculation uses Lemma 1.7.2 and the fact that $e^{-x^2y^2} \leq 1$ for all $x, y \in [0,2]$.
(ii) The result in (i) shows that for all $f \in L^1(0,2)$,

$$\int_0^2 |(Tf)(x)|\,dx \leq \int_0^2 ||f||_{L^1(0,2)}\,dx = 2\,||f||_{L^1(0,2)}.$$

Thus, T indeed maps $L^1(0,2)$ into $L^1(0,2)$ and is bounded with norm at most 2.

Problem 218:(i) This follows immediately from Corollary 10.1.4 and Theorem 7.1.2(iii).
(ii) The symmetry immediately shows that

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

On the other hand, since $xf(x)$ is an even function,

$$\int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{-2}^{0} x\,f(x)\,dx + \int_0^2 x\,f(x)\,dx = 2\int_0^2 x\,f(x)\,dx,$$

which is positive, because $xf(x) > 0$ for $x \in]0,2[$. Thus,

$$\int_{-\infty}^{\infty} x\,f(x)\,dx \neq 0.$$

By Definition 8.3.2 we conclude that f has 1 vanishing moment.