12.15 Week 12

12.15.1 Selected solutions

Exercise 10.8: (i) Direct calculation based on Corollary 10.1.4 yields that

$$\begin{aligned}\widehat{N_m}(2\gamma) = \left(\frac{1-e^{-4\pi i\gamma}}{4\pi i\gamma}\right)^m &= \left(\frac{(1-e^{-2\pi i\gamma})(1+e^{-2\pi i\gamma})}{4\pi i\gamma}\right)^m \\ &= \left(\frac{1-e^{-2\pi i\gamma}}{2\pi i\gamma}\right)^m \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^m \\ &= \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^m \widehat{N_m}(\gamma).\end{aligned}$$

Thus the condition is satisfied with

$$H_0(\gamma) := \left(\frac{1+e^{-2\pi i\gamma}}{2}\right)^m .$$

(i) H_0 is 1-periodic because the exponential function $e^{-2\pi i(\cdot)}$ is 1-periodic.

Exercise 10.11:(i)

$$G(\gamma+1) = \sum_{k\in\mathbb{Z}} |\widehat{N_m}(\gamma+1+k)|^2.$$

Letting $\ell := 1+k$, this yields that

$$G(\gamma+1) = \sum_{\ell\in\mathbb{Z}} |\widehat{N_m}(\gamma+\ell)|^2 = G(\gamma).$$

(ii)

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\gamma+k)|^2 = \sum_{k\in\mathbb{Z}} |\frac{1}{\sqrt{G(\gamma+k)}}\widehat{N_m}(\gamma+k)|^2 = \sum_{k\in\mathbb{Z}} \frac{1}{G(\gamma+k)} |\widehat{N_m}(\gamma+k)|^2.$$

By (i), we know that $G(\gamma+k) = G(\gamma)$, which is independent of k. Thus,

$$\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\gamma+k)|^2 = \frac{1}{G(\gamma)} \sum_{k\in\mathbb{Z}} |\widehat{N_m}(\gamma+k)|^2 = 1.$$

(iii) By the result in Exercise 10.8(ii), there exists a 1-periodic function H_0 such that

$$\widehat{N_m}(2\gamma) = H_0(\gamma)\widehat{N_m}(\gamma).$$

Thus,

$$\begin{aligned}
\widehat{\varphi}(2\gamma) &= \frac{1}{\sqrt{G(2\gamma)}}\,\widehat{N_m}(2\gamma) \\
&= \frac{1}{\sqrt{G(2\gamma)}} H_0(\gamma)\widehat{N_m}(\gamma) \\
&= \frac{1}{\sqrt{G(2\gamma)}} H_0(\gamma)\sqrt{G(\gamma)}\,\widehat{\varphi}(\gamma) \\
&= \sqrt{\frac{G(\gamma)}{G(2\gamma)}}\,H_0(\gamma)\,\widehat{\varphi}(\gamma).
\end{aligned}$$

Thus, the function

$$M_0(\gamma) := \sqrt{\frac{G(\gamma)}{G(2\gamma)}}\,H_0(\gamma)$$

satisfies that

$$\widehat{\varphi}(2\gamma) = M_0(\gamma)\widehat{\varphi}(\gamma).$$

The functions G and H_0 are 1-periodic, so it follows that also M_0 is 1-periodic: in fact,

$$\begin{aligned}
M_0(\gamma+1) &= \sqrt{\frac{G(\gamma+1)}{G(2(\gamma+1))}}\,H_0(\gamma+1) \\
&= \sqrt{\frac{G(\gamma+1)}{G(2\gamma+2)}}\,H_0(\gamma+1) \\
&= \sqrt{\frac{G(\gamma)}{G(2\gamma)}}\,H_0(\gamma) \\
&= M_0(\gamma).
\end{aligned}$$

(iv) We check the conditions in Theorem 8.2.11. Condition (i) follows from the fact that $\widehat{N_m}$ is a continuous function with $\widehat{N_m}(0) = 1$, together with $0 < A \le G(\gamma) \le B$. Condition (ii) is verified above, and condition (iii) is a consequence of what we proved in (i) - see Theorem 8.2.12.