12.14 Week 11

12.14.1 Selected solutions

Exercise 8.2: Direct calculation shows that

$$\hat{\phi}(\gamma) = \frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}.$$

Thus,

$$\begin{aligned}\hat{\phi}(2\gamma) = \frac{1 - e^{-4\pi i\gamma}}{4\pi i\gamma} &= \frac{\left(1 + e^{-2\pi i\gamma}\right)\left(1 - e^{-2\pi i\gamma}\right)}{4\pi i\gamma} \\ &= \frac{1 + e^{-2\pi i\gamma}}{2}\,\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma} \\ &= \frac{1 + e^{-2\pi i\gamma}}{2}\,\hat{\phi}(\gamma).\end{aligned}$$

Thus, we can take

$$H_0(\gamma) = \frac{1}{2}(1 + e^{-2\pi i\gamma}).$$

By direct calculation,

$$H_1(\gamma) = \overline{H_0(\gamma + 1/2)}e^{-2\pi i\gamma} = \cdots = -\frac{1}{2} + \frac{1}{2}\,e^{-2\pi i\gamma}.$$

Thus, using the notation in Lemma 8.2.8,

$$H_1(\gamma) = \sum d_k e^{2\pi i k\gamma},$$

where $d_{-1} = 1/2, d_0 = -1/2$. So (8.10) yields that

$$\begin{aligned}\psi(x) &= 2\left(d_{-1}\phi(2x-1) + d_0\phi(2x)\right) \\ &= -(\phi(2x) - \phi(2x-1)).\end{aligned}$$

Make a draft of this function - it is the Haar wavelet multiplied with (-1).

Exercise 8.6: Direct calculation (or a geometric argument) shows that for the Haar wavelet ψ,

$$\int_{-\infty}^{\infty} \psi(x)\,dx = 0,$$

while

$$\int_{-\infty}^{\infty} x\psi(x)\,dx \neq 0.$$

By definition, this means that the number of vanishing moments is $N = 1$.

Exercise 8.7(iii): First note that for any $j, k \in \mathbb{Z}$,

$$\psi(2^j x - k) = \begin{cases} 1 & \text{if } 2^j x - k \in [0, 1/2[, \\ -1 & \text{if } 2^j x - k \in [1/2, 1[, \\ 0 & \text{if } 2^j x - k \notin [0, 1[\end{cases} = \begin{cases} 1 & \text{if } x \in 2^{-j}[k, k+1/2[, \\ -1 & \text{if } x \in 2^{-j}[k+1/2, k+1[, \\ 0 & \text{if } x \notin 2^{-j}[k, k+1[\end{cases}.$$

Thus,

$$\begin{aligned}
d_{j,k} &= 2^{j/2}\langle f, \psi_{j,k}\rangle = 2^{j/2}\int_{-\infty}^{\infty} f(x)\overline{2^{j/2}\psi 2^j x - k}\,dx \\
&= 2^j \int_{-\infty}^{\infty} f(x)\psi(2^j x - k)\,dx = 2^j \int_{2^{-j}k}^{2^{-j}(k+1/2)} f(x)\,dx - 2^j \int_{2^{-j}(k+1/2)}^{2^{-j}(k+1)} f(x)\,dx \\
&= \frac{1}{2}\left(\frac{1}{2^{-(j+1)}}\int_{2^{-j}k}^{2^{-j}k+2^{-(j+1)}} f(x)\,dx - \frac{1}{2^{-(j+1)}}\int_{2^{-j}k+2^{-(j+1)}}^{2^{-j}k+2^{-j}} f(x)\,dx\right) \\
&= \frac{1}{2}\big(\text{average of } f \text{ over } 2^{-j}[k, k+1/2[\\
&\quad - \text{average of } f \text{ over } 2^{-j}[k+1/2, k+1[\big).
\end{aligned}$$

Exercise 8.11: (i) Note that for any $k, j \in \mathbb{Z}$, the change of variable $y = x - k$ shows that

$$\begin{aligned}
\langle T_k\phi, T_j\phi\rangle &= \int_{-\infty}^{\infty} \phi(x-k)\overline{\phi(x-j)}\,dx \\
&= \int_{-\infty}^{\infty} \phi(y)\overline{\phi(y-(j-k))}\,dx \\
&= \langle \phi, T_{j-k}\phi\rangle.
\end{aligned}$$

By definition, $\{T_k\phi\}_{k=1}^{\infty}$ is an orthonormal system if and only if

$$\langle T_k\phi, T_j\phi\rangle = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{if } k \neq 0. \end{cases}$$

By the above calculation, we conclude that this is equivalent with

$$\langle \phi, T_k\phi\rangle = \begin{cases} 1 & \text{if } k = 0; \\ 0 & \text{if } k \neq 0. \end{cases}$$

(ii) The Fourier series for the function Φ is

$$\Phi \sim \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k\gamma},$$

where

$$c_k = \int_0^1 \Phi(\gamma)e^{-2\pi i k\gamma}\,d\gamma = \langle \phi, T_{-k}\phi\rangle.$$

The function Φ is constant equal to 1 if and only if the Fourier coefficients are $c_0 = 1,\ c_k = 0,\ k \neq 0$, i.e., if and only if

$$\langle \phi, T_k\phi \rangle = \begin{cases} 1 & \text{if } k = 0; \\ 0 & \text{if } k \neq 0. \end{cases}$$

By (i) this is equivalent with $\{T_k\phi\}_{k=1}^{\infty}$ being an orthonormal system.