12.13 Week 10

12.13.1 Selected solutions

Exercise 7.9: (i)

$$\begin{aligned}\chi_{[0,1]} * \chi_{[0,2]}(y) &= \int_{-\infty}^{\infty} \chi_{[0,1]}(y-x)\chi_{[0,2]}(x)\,dx \\ &= \int_0^2 \chi_{[0,1]}(y-x)\,dx. \qquad (12.12)\end{aligned}$$

Note that

$$\chi_{[0,1]}(y-x) = \begin{cases} 1 & \text{if } y-x \in [0,1], \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in [y-1,y], \\ 0. & \text{otherwise} \end{cases}.$$

This implies that we for any given value of $y \in \mathbb{R}$ only obtain contributions to the integral (12.12) for

$$x \in [0,2] \cap [y-1,y]$$

– and for these x-values the function we integrate takes the value 1.

We calculate the integral by splitting into various intervals, namely

$$y < 0;\ 0 \le y < 1;\ 1 \le y < 2;\ 2 \le y < 3;\ 3 \le y :$$

For $y < 0$: Then

$$[0,2] \cap [y-1,y] = \emptyset,$$

and therefore

$$\chi_{[0,1]} * \chi_{[0,2]}(y) = \int_{\emptyset} dx = 0.$$

For $0 \le y \in 1$**:** Then

$$[0,2] \cap [y-1,y] = [0,y],$$

and therefore

$$\chi_{[0,1]} * \chi_{[0,2]}(y) = \int_0^y dx = y.$$

For $1 \le y \in 2$**:** Then

$$[0,2] \cap [y-1,y] = [y-1,y],$$

and therefore

$$\chi_{[0,1]} * \chi_{[0,2]}(y) = \int_{y-1}^{y} dx = 1.$$

Performing similar calculations leads to

$$\chi_{[0,1]} * \chi_{[0,2]}(y) = \begin{cases} 0 & \text{if } y < 0, \\ y & \text{if } 0 \le y < 1, \\ 1 & \text{if } 1 \le y < 2, \\ 3-y & \text{if } 2 \le y < 3, \\ 0 & \text{if } 3 \le y. \end{cases}$$

(iii) Use Theorem 7.3.4!

Exercise 8.1: (i) V_0 consists of the functions in $L^2(\mathbb{R})$ that are constant on all intervals $[k, k+1[,\ k \in \mathbb{Z}$. Since $(D^j f)(x) = 2^{j/2} f(2^j x)$,

$$V_j := D^j V_0 = \{f \in L^2(\mathbb{R}) |\ f \text{ is constant on } [2^{-j}k, 2^{-j}(k+1)[\}.$$

Make a draft of a typical function in V_{-1}, V_0, and V_1!

(ii) If $f \in V_{-1}$, then f is constant on intervals $[0,2[,\ [2,4[, ...$ Therefore f is also constant on intervals $[0,1[,\ [1,2[, [2,3[, ...$, i.e., $f \in V_0$. This shows that $V_{-1} \subseteq V_0$. The same argument shows that $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$.

Also, since $V_j = D^j V_0$, we have

$$V_{j+1} = D^{j+1} V_0 = D D^j V_0 = D V_j.$$

If $f \in V_0$ then f is constant on all intervals $[n, n+1[$. The operator T_k just translates f by k. Therefore $T_k f$ is constant on all intervals $[n, n+1[$, i.e., $T_k f \in V_0$.

Finally, direct verification shows that $\{T_k \phi\}_{k \in \mathbb{Z}}$ is an orthonormal system. In order to show that $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a orthonormal basis for V_0, we use Theorem 4.7.2. Assume that $f \in V_0$ and that $\langle f, T_k \phi \rangle = 0$ for all $k \in \mathbb{Z}$. Then

$$0 = \int_{-\infty}^{\infty} f(x) T_k \phi(x)\, dx = \int_{-\infty}^{\infty} f(x) \phi(x-k)\, dx = \int_{k}^{k+1} f(x)\, dx.$$

Since f is constant on the interval $[k, k+1[$, this implies that $f(x) = 0$ for $x \in [k, k+1[$. Since this holds for all $k \in \mathbb{Z}$, we conclude that $f(x) = 0$ for all $x \in \mathbb{R}$. According to Theorem 4.7.2 this shows that $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a orthonormal basis.

(iii) Note that by Lemma 8.2.2, $V_j = \overline{\text{span}}\{D^j T_k \phi\}_{k \in \mathbb{Z}}$. We already know that $\{T_k \phi\}_{k \in \mathbb{Z}}$ is a orthonormal basis for V_0, so by Corollary 9.1.3 we conclude that $\cap_{j \in \mathbb{Z}} V_j = \{0\}$. Also, since $\phi \in L^1(\mathbb{R})$ we know by Theorem 7.1.5 that $\hat{\phi} \in C_0(\mathbb{R})$, in particular that $\hat{\phi}$ is continuous. Since

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x)\, dx = 1,$$

we conclude by Proposition 9.4.3 that $\overline{\cup V_j} = L^2(\mathbb{R})$.

Note that the result could also be obtained using Theorem 8.2.11! However, if we do so it is not so transparent how the various results follow from the assumptions. A sketch of how the proof can be done is as follows:

Concerning condition (i) in Theorem 8.2.11, direct calculation (or Cor. 10.1.4 with $m = 1$) shows that

$$\begin{aligned} |\hat{\phi}(\gamma)| = \left|\frac{1 - e^{-2\pi i\gamma}}{2\pi i\gamma}\right| &= \left|\frac{1 - cos(2\pi\gamma) - i\sin(2\pi\gamma)}{2\pi\gamma}\right| \\ &\geq \left|\frac{\sin(2\pi\gamma)}{2\pi\gamma}\right|. \end{aligned}$$

The final expression is the absolute value of a sinc-function, which is close to 1 for γ close to 0.

Condition (ii) is a special case of Exercise 10.8 (the case m=1), and (iii) is checked above. This proves the result.