12.12 Week 9

12.12.1 Selected results

7.1: $\widehat{\chi_{[0,1]}}(\gamma) = \frac{1}{2\pi i\gamma}\,(1 - e^{-2\pi i\gamma}) = e^{-\pi i\gamma}\text{sinc}(\gamma).$

12.12.2 Selected solutions

Exercise 5.15 (i) If $f \in C_c(\mathbb{R})$, then f is bounded, i.e., there is a constant $k > 0$ such that $|f(x)| \leq k$ for all $x \in \mathbb{R}$. Since f has compact support, we have that $f(x) = 0$ outside a certain interval $[-a, a]$. With this choice of a, we conclude that $|f(x)| \leq k\,\chi_{[-a,a]}(x)$ for all $x \in \mathbb{R}$. It follows that for any $p \in [1, \infty[$,

$$\int_{-\infty}^{\infty} |f(x)|^p\,dx \leq \int_{-\infty}^{\infty} \left(k\,\chi_{[-a,a]}(x)\right)^p\,dx = k^p \int_{-a}^{a} dx = 2ak^p < \infty,$$

i.e., $f \in L^p(\mathbb{R})$.

(ii) We first consider $p = 1$. Look at the function

$$f(x) := \begin{cases} 1 & \text{if } x \in [-1,1], \\ \frac{1}{|x|} & \text{if } x \notin [-1,1]. \end{cases} \tag{12.12}$$

Then $f \in C_0(\mathbb{R})$. Since

$$\int_1^{\alpha} |f(x)|\,dx = \int_1^{\alpha} \frac{1}{x}\,dx = \ln\alpha \to \infty \text{ as } \alpha \to \infty,$$

we conclude that $f \notin L^1(\mathbb{R})$.

Now consider any $p > 1$. Then with the function f as in (12.12), the function $g(x) := (f(x))^{1/p}$ belongs to $C_0(\mathbb{R})$, but not to $L^p(\mathbb{R})$. Thus, $C_0(\mathbb{R})$ is not a subspace of $L^p(\mathbb{R})$.

Exercise 7.5: For $c > 0$,

$$\begin{aligned}(\mathcal{F}D_c f)(\gamma) &= \int_{-\infty}^{\infty} (D_c f)(x) e^{-2\pi i x\gamma}\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{c}} f(\frac{x}{c}) e^{-2\pi i x\gamma}\,dx.\end{aligned}$$

Using the change of variable $y = x/c$,

$$\begin{aligned}(\mathcal{F}D_c f)(\gamma) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{c}} f(y) e^{-2\pi i y c\gamma} c\, dy \\ &= \sqrt{c} \int_{-\infty}^{\infty} f(y) e^{-2\pi i y c \gamma}\, dy \\ &= \sqrt{c}(\mathcal{F}f)(c\gamma) \\ &= (D_{1/c}\mathcal{F}f)(\gamma).\end{aligned}$$