12.11 Week 8

12.11.1 Selected solutions

Exercise 2.6: We must show that if (2.12) holds and $\mathbf{v} \in V$, then $\mathbf{v} \in \overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty}$. Now, given $\mathbf{v} \in V$, take coefficients c_k such that $\mathbf{v} = \sum_{k=1}^{\infty} c_k \mathbf{v}_k$. By definition, this means that

$$\sum_{k=1}^{N} c_k \mathbf{v}_k \to \mathbf{v} \text{ as } N \to \infty.$$

Since $\sum_{k=1}^{N} c_k \mathbf{v}_k \in \text{span}\{\mathbf{v}_k\}_{k=1}^{\infty}$, this means by definition that $\mathbf{v} \in \overline{\text{span}}\{\mathbf{v}_k\}_{k=1}^{\infty}$.

Exercise 2.7: (ii) Using the definition of the span, we have

$$\text{span}\,\{1, x, x^2, \dots\} = \{c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N \;\big|\; N \in \mathbb{N}, c_k \in \mathbb{C}\};$$

this is exactly the set of all polynomials.

(ii) We know that any function f that can be written

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \;\; x \in]0, 1/2[\tag{12.11}$$

is infinitely often differentiable. Thus,

$$f(x) := \begin{cases} x & \text{if } x \in [0, 1/4], \\ \frac{1}{2} - x & \text{if } x \in [1/4, 1/2] \end{cases}$$

is an example of a continuous function that can not be written on the form (12.11).

Exercise 3.6: By definition, $\text{span}\{\delta_k\}_{k=1}^{\infty}$ is the set of *finite* linear combinations of the vectors δ_k. This is exactly the set of sequences which only have a finite number of nonzero coordinates - and this is exactly the vector space V considered in Exercise 3.5 and Lemma 3.2.4.

Problem 220: (ii) The function f is not bounded: for each integer $k \geq 2$ we have $f(x) = k$ for $x \in [k, k + k^{-3}]$, so $\sup_{x \in \mathbb{R}} |f(x)| = \infty$.

(iii) We shall check whether $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. In order to do so, let $N \in \mathbb{N}, N \geq 2$. Then

$$\int_{-N}^{N} |f(x)|\, dx = \int_{-N}^{N} |\sum_{k=2}^{\infty} k\, \chi_{[k, k+k^{-3}]}(x)|\, dx = \int_{-N}^{N} \sum_{k=2}^{\infty} k\, \chi_{[k, k+k^{-3}]}(x)\, dx.$$

Since the ingegral goes from $-N$ to N the terms in the infinite series are only contributing for $k = 2, 3, \dots, N-1$. Thus,

$$\int_{-N}^{N} |f(x)|\,dx = \int_{-N}^{N} \sum_{k=2}^{N-1} k\,\chi_{[k,k+k^{-3}]}(x)\,dx$$
$$= \sum_{k=2}^{N-1} k \int_{-N}^{N} \chi_{[k,k+k^{-3}]}(x)\,dx = \sum_{k=2}^{N-1} k\,\frac{1}{k^3} = \sum_{k=2}^{N-1} \frac{1}{k^2}.$$

This shows that $\int_{-N}^{N} |f(x)|\,dx$ has a limit for $N \to \infty$ (namely, $\sum_{k=2}^{\infty} \frac{1}{k^2} < \infty$). By definition this means that $f \in L^1(\mathbb{R})$.

Problem 223: (i) For any $\alpha, \beta \in \mathbb{C}$ and any functions $f, g : \mathbb{R} \to \mathbb{C}$,

$$E_b(\alpha f + \beta g)(x) = e^{2\pi i b x}\,(\alpha f(x) + \beta g(x))$$
$$= \alpha e^{2\pi i b x} f(x) + \beta e^{2\pi i b x} g(x) = \alpha E_b f(x) + \beta E_b g(x).$$

We conclude that E_b is linear.

(ii) Writing $b = \alpha + i\beta,\ \alpha, \beta \in \mathbb{R}$,

$$\int_0^1 |E_b f(x)|^2\,dx = \int_0^1 |e^{2\pi i b x} f(x)|^2\,dx$$
$$= \int_0^1 e^{-4\pi\beta x} |f(x)|^2\,dx$$
$$\le \left(\sup_{x\in[0,1]} e^{-4\pi\beta x}\right) \int_0^1 |f(x)|^2\,dx = \sup_{x\in[0,1]} e^{-4\pi\beta x}\,||f||^2.$$

Since $C := \sup_{x\in[0,1]} e^{-4\pi\beta x} < \infty$, we conclude that $E_b f \in L^2(0,1)$, i.e., the operator E_b maps $L^2(0,1)$ into $L^2(0,1)$ and is hence well defined. The calculation also shows that

$$||E_b f|| \le \sqrt{C}\,||f||,$$

i.e., E_b is bounded.

(iii) Writing $b = \alpha + i\beta,\ \alpha, \beta \in \mathbb{R}$,

$$E_b f(x) = e^{2\pi i b x}\, f(x) = e^{2\pi i \alpha x} e^{-2\pi\beta x}\, f(x).$$

Thus, for any $f, g \in L^2(0,1)$,

$$\langle E_b f, g\rangle = \int_0^1 e^{2\pi i b x}\, f(x)\overline{g(x)}\,dx = \int_0^1 e^{2\pi i \alpha x} e^{-2\pi\beta x}\, f(x)\overline{g(x)}\,dx$$
$$= \int_0^1 f(x)\overline{e^{-2\pi i \alpha x} e^{-2\pi\beta x} g(x)}\,dx.$$

This shows that

$$E_b^* g(x) = e^{-2\pi i \alpha x} e^{-2\pi\beta x} g(x).$$

Since

$$E_b g(x) = e^{2\pi i \alpha x} e^{-2\pi \beta x} g(x),$$

this shows that $E_b = E_b^*$ if and only if $\alpha = 0$. That is, the operator E_b is self-adjoint if and only if b has the form

$$b = i\beta, \;\; \beta \in \mathbb{R}.$$