12.10 Week 7

12.10.1 Selected solutions

Problem 219:(i) Using Theorem 4.7.2,

$$\sum_{k=1}^{\infty} |\langle \mathbf{v}, T\mathbf{e}_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle T^*\mathbf{v}, \mathbf{e}_k\rangle|^2 = ||T^*\mathbf{v}||^2.$$

(ii) By Exercise 4.17(ii) we have

$$T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I,$$

and similarly $(T^{-1})^*T^* = I$. By Definition 2.5.4 this means that T^* is invertible and that $(T^*)^{-1} = (T^{-1})^*$.
(iii) Since $\mathbf{v} = (T^*)^{-1}T^*\mathbf{v}$,

$$||\mathbf{v}|| = ||(T^*)^{-1}T^*\mathbf{v}|| \le ||(T^*)^{-1}||\, ||T^*\mathbf{v}||.$$

(iv) By the result in (i),

$$\sum_{k=1}^{\infty} |\langle \mathbf{v}, T\mathbf{e}_k\rangle|^2 = ||T^*\mathbf{v}||^2 \le ||T^*||^2\, ||f||^2.$$

The upper inequality now follows from Lemma 4.5.2 (ii).

To prove the lower inequality, note that by (iii), followed by an application of (ii) and Lemma 4.5.2,

$$||T^*\mathbf{v}||^2 \ge \frac{1}{||(T^*)^{-1}||^2}\, ||\mathbf{v}||^2 = \frac{1}{||(T^{-1})^*||^2}\, ||\mathbf{v}||^2 = \frac{1}{||T^{-1}||^2}\, ||\mathbf{v}||^2.$$

The result now again follows from (i).

Exercise 4.31: We want to use the result in Exercise 4.25, so we first show that $\sum_{k=1}^{\infty} c_k\mathbf{e}_k$ is convergent if $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$. The idea is to show that the sequence $\{\sum_{k=1}^{n} c_k\mathbf{e}_k\}_{n=1}^{\infty}$ is a Cauchy sequence. Given any $m, n \in \mathbb{N},\ n \ge m$,

$$\begin{aligned} \left|\left|\sum_{k=1}^{n} c_k\mathbf{e}_k - \sum_{k=1}^{m} c_k\mathbf{e}_k\right|\right|^2 &= \left|\left|\sum_{k=m+1}^{n} c_k\mathbf{e}_k\right|\right|^2 \\ &= \langle \sum_{k=m+1}^{n} c_k\mathbf{e}_k, \sum_{j=m+1}^{n} c_j\mathbf{e}_j\rangle = \sum_{k=m+1}^{n}\sum_{j=m+1}^{n} c_k\overline{c_j}\langle \mathbf{e}_k, \mathbf{e}_j\rangle. \end{aligned}$$

Using that $\{\mathbf{e}_k\}_{k=1}^{\infty}$ is an orthonormal system, we only get contributions for $j = k$, and arrive at

$$\left|\left|\sum_{k=1}^{n} c_k\mathbf{e}_k - \sum_{k=1}^{m} c_k\mathbf{e}_k\right|\right|^2 = \sum_{k=m+1}^{n} |c_k|^2.$$

Since $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$ this implies that $\{\sum_{k=1}^{n} c_k \mathbf{e}_k\}_{n=1}^{\infty}$ is indeed a Cauchy sequence, and therefore convergent. Now where we know that $\sum_{k=1}^{\infty} c_k \mathbf{e}_k$ is convergent, we can use the result in Exercise 4.25 to repeat the above calculation with an *infinite* sum:

$$\begin{aligned}\left|\left|\sum_{k=1}^{\infty} c_k \mathbf{e}_k\right|\right|^2 &= \langle \sum_{k=1}^{\infty} c_k \mathbf{e}_k, \sum_{j=1}^{\infty} c_j \mathbf{e}_j \rangle \\ &= \sum_{j=1}^{\infty} \overline{c_j} \langle \sum_{k=1}^{\infty} c_k \mathbf{e}_k, \mathbf{e}_j \rangle \quad \text{(by Exercise 4.25)} \\ &= \sum_{k=1}^{\infty} c_k \overline{c_j} \langle \mathbf{e}_k, \mathbf{e}_j \rangle \quad \text{(by Exercise 4.25)} \\ &= \sum_{k=1}^{\infty} |c_k|^2.\end{aligned}$$

Problem 107

(i) Since

$$\sum_{n=-\infty}^{\infty} |d_n e^{inx}| = \sum_{n=-\infty}^{\infty} |d_n|, \ \forall x \in \mathbb{R},$$

the series $\sum_{n=-\infty}^{\infty} |d_n|$ is a convergent majorant series.

(ii) For $m = n$,

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \int_{-\pi}^{\pi} dx = 2\pi,$$

and for $m \neq n$,

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = \int_{-\pi}^{\pi} e^{ix(n-m)} dx = \left[\frac{1}{i(n-m)} e^{ix(n-m)} \right]_{x=-\pi}^{\pi} = 0.$$

(iii) For any $m \in \mathbb{N}$, interchanging the order of integration and summation (you need to justify that this is allowed!), yields that

$$\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} d_n e^{inx} e^{-imx}\, dx = \sum_{n=-\infty}^{\infty} d_n \int_{-\pi}^{\pi} e^{inx} e^{-imx}\, dx = 2\pi d_m.$$

(iv) Inserting that $f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx}$ in the result in (iii) yields that

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx = 2\pi d_m,$$

and the desired formula is obtained by a division by 2π.