12.8 Week 6

12.8.1 Selected results

12.8.2 Selected solutions

Exercise 5.9:

(i) Example 5.2.3 shows that the function $f(x) = \frac{1}{2\sqrt{x}}\chi_{]0,1]}(x)$ belongs to $L^1(\mathbb{R})$. Since

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_0^1 \frac{1}{4x}\,dx = \frac{1}{4}\,[\ln x]_{x\to 0}^{1} = \infty,$$

we see that $f \notin L^2(\mathbb{R})$; thus, $L^1(\mathbb{R})$ is not contained in $L^2(\mathbb{R})$.

(ii) Consider the function f in Example 5.1.2(iii). Since

$$\int_1^{\alpha} \frac{1}{|x|}\,dx = \ln\alpha \to \infty \text{ as } \alpha \to \infty,$$

we see that $f \notin L^1(\mathbb{R})$. We now show that $f \in L^2(\mathbb{R})$. First,

$$\int_1^{\alpha} |f(x)|^2\,dx = \int \frac{1}{x^2}\,dx = \left[\frac{-1}{\alpha}\right]_1^{\alpha} = 1 - \frac{1}{\alpha} \to 1 \text{ as } \alpha \to \infty;$$

thus,

$$\int_1^{\infty} |f(x)|^2\,dx = 1.$$

By symmetry,

$$\int_{-\infty}^{2} |f(x)|^2\,dx = 1,$$

so

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|^2\,dx &= \int_{-\infty}^{-1} |f(x)|^2\,dx + \int_{-1}^{1} |f(x)|^2\,dx + \int_1^{\infty} |f(x)|^2\,dx \\ &= 2 + \int_{-1}^{1} 1\,dx = 4 < \infty;\end{aligned}$$

we conclude that $f \in L^2(\mathbb{R})$. This shows that $L^2(\mathbb{R})$ is not contained in $L^1(\mathbb{R})$.

(iii) The assumption that f has compact support implies that for some $N \in \mathbb{N}$,

$$\operatorname{supp} f \subseteq [-N, N].$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} |f(x)|\,dx &= \int_{-N}^{N} |f(x)|\,dx \\ &= \int_{-N}^{N} 1\cdot|f(x)|\,dx \\ &\le \left(\int_{-N}^{N} 1^2\,dx\right)^{1/2} \left(\int_{-N}^{N} |f(x)|^2\,dx\right)^{1/2} \\ &= \sqrt{2N}\left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^{1/2},\end{aligned}$$

which is finite if $f \in L^2(\mathbb{R})$. So $f \in L^1(\mathbb{R})$.

Exercise 6.1:(ii) Since $f_n(x) = f(x) = 1$ for $x \in [1, 2]$,

$$\begin{aligned}||f - f_n||^2 &= \int_0^2 |f(x) - f_n(x)|^2\,dx \\ &= \int_0^1 |f(x) - f_n(x)|^2\,dx \\ &= \int_0^1 x^{2n}\,dx \\ &= \frac{1}{2n+1} \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

(iii) The result in (ii) shows that the sequence $\{f_k\}_{k=1}^{\infty}$ consisting of functions in $C[0, 2]$ converges to the function f. But f is not continuous, i.e., $f \notin C[0, 2]$. Thus, within the space $C[0, 2]$ the sequence $\{f_k\}_{k=1}^{\infty}$ is not convergent. This proves that $C[0, 2]$ cannot be a Hilbert space with respect to the norm considered here.

12.9 Examples and slides from the lecture

Example 12.9.1 In quantum mechanics the *momentum operator* plays a key role. For a certain real constant $\hbar > 0$ (Plancks constant divided by 2π, $\hbar \approx 1.05 \cdot 10^{-34}$) it has the form

$$P : L^2(\mathbb{R}) \to L^2(\mathbb{R}), Pf := i\hbar \frac{df}{dx}.$$

Since $L^2(\mathbb{R})$ contains functions that are not necessarily differentiable (for example $\chi_{[0,1]}$), the operator P is not well defined. And - even if $f \in L^2(\mathbb{R})$ is differentiable, it is not clear that the derivative will belong to $L^2(\mathbb{R})$. We will therefore consider the subspace of $L^2(\mathbb{R})$ given by

$$V := \{f \in L^2(\mathbb{R}) \cap C_0(\mathbb{R}) \mid f \text{ is differentiable and } \frac{df}{dx} \in L^2(\mathbb{R})\}.$$

Then the operator

$$P : V \to L^2(\mathbb{R}), Pf := i\hbar \frac{df}{dx}$$

is well defined. It is clear that P is linear. Note that the assumption $f \in C_0(\mathbb{R})$ is not necessary for this part (but we will need it later in the example). We will now show that P is unbounded. For this purpose, consider the functions $f_k,\ k \in \mathbb{N}$, given by

$$f_k(x) := \begin{cases} \frac{3}{2} - \frac{1}{2}k^2x^2, & x \in [-1/k, 1/k], \\ e\, e^{-k\,|x|}, & x \notin [-1/k, 1/k]. \end{cases}$$

Direct verification shows that $f_k \in V$ (in fact the construction is made to ensure that f_k is differentiable at the points $\pm 1/k$). Also,

$$f_k'(x) := \begin{cases} ke^{-k\,|x|}, & x \in]-\infty, 1/k], \\ -k^2x, & x \in [-1/k, 1/k], \\ -ke\, e^{-k\,|x|}, & x \in [1/k, \infty[. \end{cases}$$

For symmetry reasons,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_k(x)|^2\,dx &= 2\int_0^{\infty} |f_k(x)|^2\,dx \\
&= 2\left(\int_0^{1/k} \left(\frac{3}{2} - \frac{1}{2}k^2x^2\right)^2 dx + \int_{1/k}^{\infty} (e\,e^{-k\,x})^2\,dx\right) \\
&= 2\left(\int_0^{1/k} (\frac{9}{4} + \frac{1}{4}k^4x^4 - \frac{3}{2}k^2x^2)\,dx + \int_{1/k}^{\infty} e^2\,e^{-2k\,x}\,dx\right) \\
&= 2\left(\frac{9}{4}\frac{1}{k} + \frac{1}{20}\frac{1}{k} + \frac{3}{6}\frac{1}{k} + e^2\,\frac{1}{2k}e^{-2}\right) = \frac{33}{5}\,\frac{1}{k}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{-\infty}^{\infty} |f_k'(x)|^2\,dx &= 2\int_0^{\infty} |f_k'(x)|^2\,dx\\ &= 2\left(\int_0^{1/k} (-k^2 x)^2\,dx + \int_{1/k}^{\infty} (ke\,e^{-k\,x})^2\,dx\right)\\ &= 2\left(k^4\frac{1}{3k^3} - k^2e^2\frac{1}{-2k}e^{-2}\right)\\ &= \frac{5}{3}\,k.\end{aligned}$$

It follows from these calculations that

$$||Pf_k||_2 = \left(\int_{-\infty}^{\infty} |i\hbar f_k'(x)|^2\,dx\right)^{1/2} = \hbar\sqrt{\frac{5}{3}\,k}.$$

In order to show that P is unbounded, we will now relate the quantities $||Pf_k||_2$ and $||f_k||_2$:

$$||Pf_k||_2 = \hbar\sqrt{\frac{5}{3}\,k} = \hbar\sqrt{\frac{\frac{5}{3}\,k}{\frac{33}{5}\,\frac{1}{k}}}\,||f_k||_2 = k\hbar\frac{5}{\sqrt{99}}\,k||f_k||_2.$$

Since $k\hbar\,\frac{5}{\sqrt{99}} \to \infty$ as $k \to \infty$, we conclude that P is unbounded.

Even though the operator P is unbounded, it turns out to have a property that is similar to the self-adjoint bounded operators we hace considered in this course. In fact, for any $f, g \in V$, partial integration shows that

$$\begin{aligned}\langle Pf, g\rangle &= \int_{-\infty}^{\infty} (Pf)(x)\overline{g(x)}\,dx = i\hbar\int_{-\infty}^{\infty} \frac{df}{dx}(x)\overline{g(x)}\,dx\\ &= i\hbar\left(\Big[f(x)\overline{g(x)}\Big]_{x\to-\infty}^{x\to\infty} - \int_{-\infty}^{\infty} f(x)\overline{\frac{dg}{dx}(x)}\,dx\right).\end{aligned}$$

The first term vanishes because of the assumption $f, h \in C_0(\mathbb{R})$. Thus,

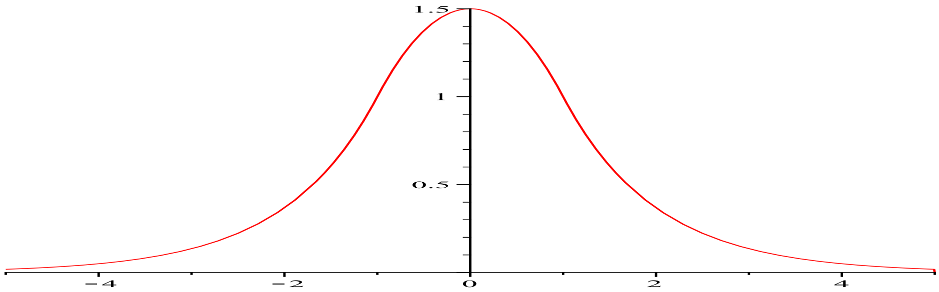
$$\langle Pf, g\rangle = -i\hbar\int_{-\infty}^{\infty} f(x)\overline{\frac{dg}{dx}(x)}\,dx = \int_{-\infty}^{\infty} f(x)\overline{i\hbar\frac{dg}{dx}(x)}\,dx = \langle f, Pg\rangle.$$

In less stringent literature it is often claimed that P is a self-adjoint operator because of this, even though it is (as we have seen) not a bounded operator.

The figure on the next page shows the functions f_1 and f_5. The reader might wonder where the idea of the above calculation comes from. In order to understand this, note that for $x \neq 0$, the functions $g_k(x) = ee^{-k|x|}$, are differentiable, and

$$|g_k'(x)| = k\, g_k(x).$$

Thus, the L^2-norm of g_k is increased by the factor k by the differential operator $\frac{d}{dx}$. Since $k \in \mathbb{N}$ is arbitrary this will imply that the operator P is unbounded. However, this argument is not completely precise yet, because the functions g_k are not differentiable at zero! The functions f_k in the example are obtained by modifying g_k around $x = 0$ so it becomes differentiable everywhere; hereby a rigorous proof is obtained. The two last figures show the functions g_1 and g_5.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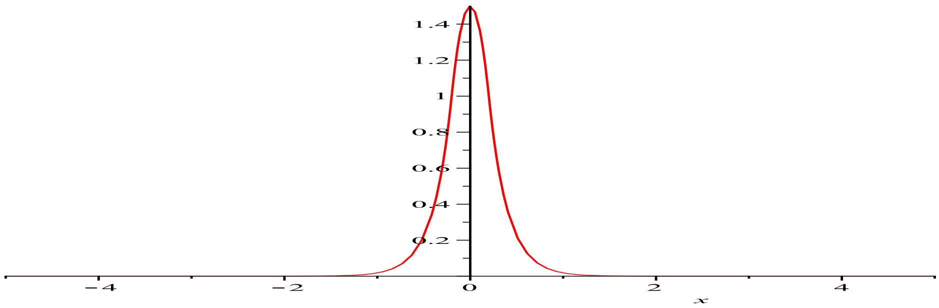
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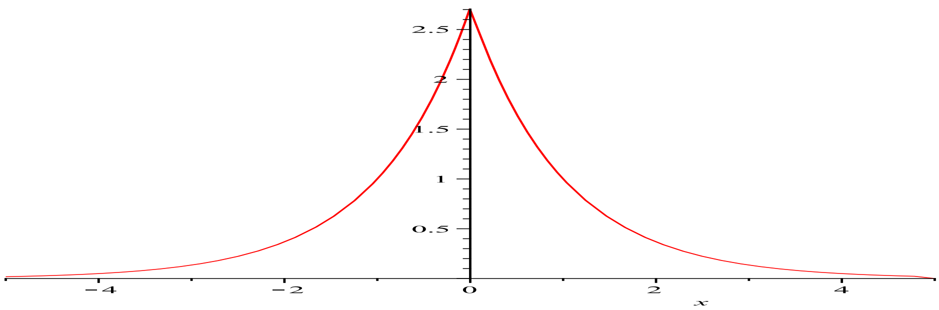
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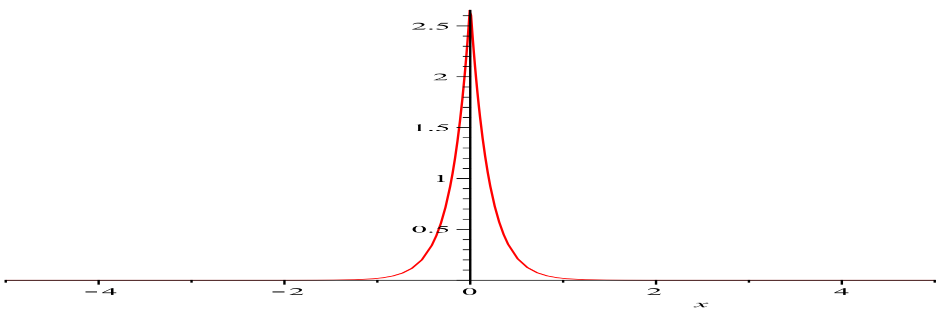
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Key properties of central normed spaces of functions:

(A) - The space

$$C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ cont. with compact support}\}.$$

(i) The assumption of compact support is realistic for applications;

(ii) Unfortunately not all signals are continuous;

(iii) Unfortunately $C_c(\mathbb{R})$ is not a Banach space.

(B) - The space

$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \text{ cont., } f(x) \to 0 \text{ as } x \to \pm\infty\}.$$

(i) $C_o(\mathbb{R})$ is a Banach space.

(ii) The assumption $f(x) \to 0$ as $x \to \pm\infty$ is realistic for applications;

(iii) Unfortunately not all signals are continuous;

(C) - the space $L^1(\mathbb{R})$

(i) $L^1(\mathbb{R})$ is a Banach space.

(ii) Contains functions that are not continuous

(iii) Realistic assumption for applications;

(iv) Unfortunately $L^1(\mathbb{R})$ is not a Hilbert space;

(v) Unfortunately $L^1(\mathbb{R})$ is really a space of equivalence classes.

(D) - the space $L^2(\mathbb{R})$

(i) $L^2(\mathbb{R})$ is a Banach space.

(ii) Contains functions that are not continuous

(iii) Realistic assumption for applications;

(iv) $L^2(\mathbb{R})$ is a Hilbert space;

(v) Unfortunately $L^2(\mathbb{R})$ is really a space of equivalence classes.