12.6 Week 5

12.6.1 *Selected results*

Problem 4: (ii) $\text{Supp}f_3 = [-1, 2]$, $\text{Supp}f_4 = [-1, 1]$; (iii) f_4, f_5; (iv) f_4; (v) f_3, f_4, f_5.

12.6.2 *Selected solutions*

Exercise 5.4:(i) We check that $\langle f, f\rangle = 0 \Rightarrow f(x) = 0$ for all $x \in \mathbb{R}$ (the rest of the conditions in Definition 4.1.1 follow by direct verification). In order to do that, assume that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. Then $|f(x_0)|^2 > 0$, and the continuity of f implies that

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx > 0.$$

Thus $\langle f, f\rangle > 0$, which is a contradiction.

(ii) In order to show that $C_c(\mathbb{R})$ does not form a Hilbert space, it is enough to find *one* Cauchy sequence $\{f_k\}_{k=1}^{\infty}$ in $C_c(\mathbb{R})$ that does not converges to a function in $C_c(\mathbb{R})$. We could have used the functions from the proof of Lemma 5.1.4, but we use a different collection of functions below.Let

$$f_k(x) := \begin{cases} 1 - x \text{ if } x \in [0, 1], \\ 1 + kx \text{ if } x \in [-1/k, 0[, \\ 0 \text{ if } x \notin [-1/k, 1], \end{cases}$$

and

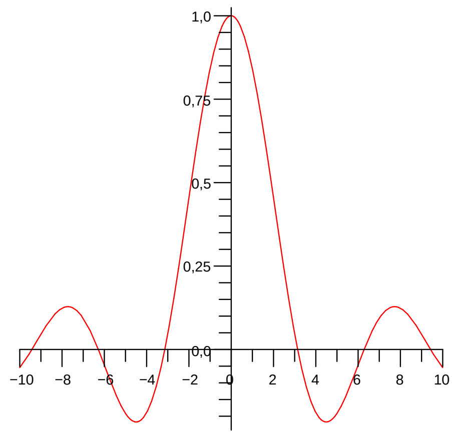
$$f(x) := \begin{cases} 1 - x \text{ if } x \in [0, 1], \\ 0 \text{ if } x \notin [0, 1], \end{cases}$$

Then $f_k \in C_c(\mathbb{R})$ for all $k \in \mathbb{N}$ (make a draft), but f is not continuous, so $f \notin C_c(\mathbb{R})$. Now, using the norm $|| \cdot ||_2$ associated with the given inner product,

$$\begin{aligned}
||f - f_k||_2^2 &= \int_{-\infty}^{\infty} |f(x) - f_k(x)|^2 \, dx \\
&= \int_{-1/n}^{0} (1 + kx)^2 \, dx \\
&= \left[x + kx^2 + \frac{1}{3} k^2 x^3 \right]_{x=-1/k}^{0} \\
&= \frac{1}{3} \frac{1}{k} \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

Thus, the sequence $\{f_k\}_{k=1}^{\infty}$ in $C_c(\mathbb{R})$ converges to the function f. By Lemma 3.1.3 this implies that $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence. However, the limit, i.e., the function f, does not belong to $C_c(\mathbb{R})$! Thus $C_c(\mathbb{R})$ does not form a Banach space with respect to the norm arising from the inner product.

12.7 Examples and slides from the lecture



The function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$