12.4 Week 4

12.4.1 Selected results

4.21: $T^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}),\ T^*(y_1, y_2, y_3, y_4, \dots) = (2y_2, 3y_1, y_3, y_4, \dots).$

12.4.2 Selected solutions

Exercise 4.18: (i) By definition of the adjoint operator and the rules for calculation with the inner product,

$$\begin{aligned}\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T^*\mathbf{w}\rangle = \overline{\langle T^*\mathbf{w}, \mathbf{v}\rangle} &= \overline{\langle \mathbf{w}, (T^*)^*\mathbf{v}\rangle} \\ &= \langle (T^*)^*\mathbf{v}, \mathbf{w}\rangle.\end{aligned}$$

This holds for all $\mathbf{v}, \mathbf{w} \in \mathcal{H}$. By Lemma 4.4.2, this implies that

$$T\mathbf{v} = (T^*)^*\mathbf{v}$$

for all $\mathbf{v} \in \mathcal{H}$, i.e., that $T = (T^*)^*$.

(ii) By (4.15), we know that

$$||T^*|| \le ||T||. \tag{12.9}$$

Applying this to the operator T^* yields that $||(T^*)^*|| \le ||T^*||$; However, by (i) we know that $(T^*)^* = T$; thus, we have shown that $||T|| \le ||T^*||$. Together with (12.9) this yields the result.

Exercise 4.31: We want to use the result in Exercise 4.25, so we first show that $\sum_{k=1}^\infty c_k\mathbf{e}_k$ is convergent if $\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$. The idea is to show that the sequence $\{\sum_{k=1}^n c_k\mathbf{e}_k\}_{n=1}^\infty$ is a Cauchy sequence. Given any $m, n \in \mathbb{N},\ n \ge m$,

$$\begin{aligned}\left|\left|\sum_{k=1}^n c_k\mathbf{e}_k - \sum_{k=1}^m c_k\mathbf{e}_k\right|\right|^2 &= \left|\left|\sum_{k=m+1}^n c_k\mathbf{e}_k\right|\right|^2 \\ &= \langle \sum_{k=m+1}^n c_k\mathbf{e}_k, \sum_{j=m+1}^n c_j\mathbf{e}_j\rangle \\ &= \sum_{k=m+1}^n \sum_{j=m+1}^n c_k\overline{c_j}\langle \mathbf{e}_k, \mathbf{e}_j\rangle.\end{aligned}$$

Using that $\{\mathbf{e}_k\}_{k=1}^\infty$ is an orthonormal system, we only get contributions for $j = k$, and arrive at

$$\left|\left|\sum_{k=1}^n c_k\mathbf{e}_k - \sum_{k=1}^m c_k\mathbf{e}_k\right|\right|^2 = \sum_{k=m+1}^n |c_k|^2.$$

Since $\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$ this implies that $\{\sum_{k=1}^n c_k\mathbf{e}_k\}_{n=1}^\infty$ is indeed a Cauchy sequence, and therefore convergent. Now where we know that $\sum_{k=1}^\infty c_k\mathbf{e}_k$ is convergent, we can use the result in Exercise 4.25 to repeat the above calculation with an *infinite* sum:

$$\begin{aligned}\left|\left|\sum_{k=1}^\infty c_k\mathbf{e}_k\right|\right|^2 &= \langle\sum_{k=1}^\infty c_k\mathbf{e}_k, \sum_{j=1}^\infty c_j\mathbf{e}_j\rangle \\ &= \sum_{j=1}^\infty \overline{c_j}\langle\sum_{k=1}^\infty c_k\mathbf{e}_k, \mathbf{e}_j\rangle \quad \text{(by Exercise 4.25)} \\ &= \sum_{j=1}^\infty\sum_{k=1}^\infty c_k\overline{c_j}\langle\mathbf{e}_k, \mathbf{e}_j\rangle \quad \text{(by Exercise 4.25)} \\ &= \sum_{k=1}^\infty |c_k|^2.\end{aligned}$$

12.5 Examples and slides from the lecture

Example 12.5.1 Consider the mapping

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}),\ T\{x_k\}_{k=1}^\infty := \{x_k + x_{k+1}\}_{k=1}^\infty.$$

From the example in weeklynote2.pdf we know that T is linear and bounded. Let us calculate the adjoint T^*, which satisfies that

$$\langle T\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty\rangle = \langle\{x_k\}_{k=1}^\infty, T^*\{y_k\}_{k=1}^\infty\rangle. \tag{12.10}$$

for all $\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty \in \ell^2(\mathbb{N})$. The idea is to "isolate $\{x_k\}_{k=1}^\infty$," i.e., to "isolate $x_1, x_2, \cdots :$"

$$\begin{aligned}\langle T\{x_k\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty\rangle &= \langle\{x_k + x_{k+1}\}_{k=1}^\infty, \{y_k\}_{k=1}^\infty\rangle \\ &= \sum_{k=1}^\infty (x_k + x_{k+1})\overline{y_k} \\ &= (x_1 + x_2)\overline{y_1} + (x_2 + x_3)\overline{y_2} + \cdots \\ &= x_1\overline{y_1} + x_2\overline{(y_1 + y_2)} + x_3\overline{(y_3 + y_4)} + \cdots \\ &= \langle\left\{\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \cdots \\ \cdots \end{array}\right\}, \left\{\begin{array}{c} y_1 \\ y_1 + y_2 \\ y_2 + y_3 \\ \cdots \\ \cdots \end{array}\right\}\rangle\end{aligned}$$

By (12.10) this shows that

$$T^*(y_1, y_2, \dots) = (y_1, y_1 + y_2, y_2 + y_3, \dots).$$

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