## 12.3 Week 3

### *12.3.1 Selected solutions*

**Exercise 4.4:** Let $\mathbf{v} \in \mathcal{H}$. Then, for any $\mathbf{w} \in \mathcal{H}$ with $||\mathbf{w}|| = 1$, Cauchy–Schwarz' inequality shows that

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}|| \, ||\mathbf{w}|| = ||\mathbf{v}||;$$

thus,

$$\sup_{||\mathbf{w}||=1} |\langle \mathbf{v}, \mathbf{w} \rangle| \leq ||\mathbf{v}||. \tag{12.4}$$

If $\mathbf{v} = \mathbf{0}$, the stated result clearly holds, so let us assume that $\mathbf{v} \neq \mathbf{0}$. Put $\mathbf{w} = \frac{\mathbf{v}}{||\mathbf{v}||}$. Then $||\mathbf{w}|| = 1$, and

$$|\langle \mathbf{v}, \mathbf{w} \rangle| = |\langle \mathbf{v}, \frac{\mathbf{v}}{||\mathbf{v}||} \rangle| = \frac{1}{||\mathbf{v}||} \langle \mathbf{v}, \mathbf{v} \rangle = ||\mathbf{v}||.$$

This implies that

$$\sup_{||\mathbf{w}||=1} |\langle \mathbf{v}, \mathbf{w} \rangle| \geq ||\mathbf{v}||. \tag{12.5}$$

Put together, the two inequalities (12.4) and (12.5) prove the result.

**Exercise 4.9:**(i) By definition, $||\mathbf{e}_1|| = ||\mathbf{e}_2|| = 1$. Furthermore,

$$\begin{aligned}
\langle \mathbf{e}_2, \mathbf{e}_1 \rangle &= \langle \frac{\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1}{||\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1||}, \mathbf{e}_1 \rangle \\
&= \frac{1}{||\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1||} (\langle \mathbf{v}_2, \mathbf{e}_1 \rangle - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \langle \mathbf{e}_1, \mathbf{e}_1 \rangle) \\
&= 0.
\end{aligned}$$

By definition,

$$\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\};$$

since both spaces have dimension equal to two, it follows that

$$\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

(ii) Since $\{\mathbf{v}_k\}_{k=1}^{n+1}$ are linearly independent,

$$\mathbf{v}_{n+1} \notin \text{span}\{\mathbf{v}_k\}_{k=1}^{n} = \text{span}\{\mathbf{e}_k\}_{k=1}^{n}.$$

This implies that

$$\mathbf{v}_{n+1} \neq \sum_{k=1}^{n} \langle \mathbf{v}_{n+1}, \mathbf{e}_k \rangle \mathbf{e}_k.$$

(iii) We now proceed with an induction argument. Assume that for some $n \in \mathbb{N}$, the family $\{\mathbf{e}_k\}_{k=1}^n$ is constructed via the stated formula, that $\{\mathbf{e}_k\}_{k=1}^n$ is an orthonormal system, and that

$$\text{span}\{\mathbf{e}_k\}_{k=1}^n = \text{span}\{\mathbf{v}_k\}_{k=1}^n.$$

With $\mathbf{e}_{n+1}$ given by the stated formula, we have that $||\mathbf{e}_{n+1}|| = 1$; furthermore, for each $\ell = 1, \dots, n$,

$$\begin{aligned} & \langle \mathbf{e}_{n+1}, \mathbf{e}_\ell \rangle \\ = \ & \langle \frac{\mathbf{v}_{n+1} - \sum_{k=1}^n \langle \mathbf{v}_{n+1}, \mathbf{e}_k \rangle \mathbf{e}_k}{||\mathbf{v}_{n+1} - \sum_{k=1}^n \langle \mathbf{v}_{n+1}, \mathbf{e}_k \rangle \mathbf{e}_k||}, \mathbf{e}_\ell \rangle \\ = \ & \frac{1}{||\mathbf{v}_{n+1} - \sum_{k=1}^n \langle \mathbf{v}_{n+1}, \mathbf{e}_k \rangle \mathbf{e}_k||} \left( \langle \mathbf{v}_{n+1}, \mathbf{e}_\ell \rangle - \sum_{k=1}^n \langle \mathbf{v}_{n+1}, \mathbf{e}_k \rangle \langle \mathbf{e}_k \mathbf{e}_\ell \rangle \right) \\ = \ & 0. \end{aligned}$$

This shows that $\{\mathbf{e}_k\}_{k=1}^{n+1}$ is an orthonormal system. Since

$$\text{span}\{\mathbf{e}_k\}_{k=1}^{n+1} \subseteq \text{span}\{\mathbf{v}_k\}_{k=1}^{n+1}$$

and

$$\dim\left(\text{span}\{\mathbf{e}_k\}_{k=1}^{n+1}\right) = \dim\left(\text{span}\{\mathbf{v}_k\}_{k=1}^{n+1}\right) = n+1,$$

we finally obtain that

$$\text{span}\{\mathbf{e}_k\}_{k=1}^{n+1} = \text{span}\{\mathbf{v}_k\}_{k=1}^{n+1}$$

(iv) The result in (iii) together with the definition of the span of an *infinite* collection of vectors show that

$$\text{span}\{\mathbf{e}_k\}_{k=1}^{\infty} = \text{span}\{\mathbf{v}_k\}_{k=1}^{\infty}.$$

Therefore also the closure of these two sets are identical.
**Exercise 4.12:** Assume that $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v} \in \mathcal{H}$. Then $\langle \mathbf{v}, \mathbf{u} - \mathbf{w} \rangle = 0$ for all $\mathbf{v} \in \mathcal{H}$. Taking $\mathbf{v} =: \mathbf{u} - \mathbf{w}$ this shows that

$$\langle \mathbf{u} - \mathbf{w}, \mathbf{u} - \mathbf{w} \rangle = 0.$$

By definition of the inner product this implies that $\mathbf{u} - \mathbf{w} = \mathbf{0}$, i.e., that $\mathbf{u} = \mathbf{w}$.

### *12.3.2 Examples and slides from the lecture*

**Example 12.3.1** Let $\mathcal{H}$ be a Hilbert space and $\{\mathbf{v}_k\}_{k=1}^{\infty}$ a sequence of vectors in $\mathcal{H}$ with $||\mathbf{v}_k|| = 1$. Consider the mapping

$$\Phi : \mathcal{H} \to \mathbb{C},\ \Phi\mathbf{v} := \sum_{k=1}^{\infty} \frac{1}{k^2}\, \langle \mathbf{v}, \mathbf{v}_k \rangle.$$

We want to show the following that $\Phi$ is a bounded functional on $\mathcal{H}$, and proceed by the following steps. We show

(i) That $\Phi$ is well defined, i.e., that $\sum_{k=1}^{\infty} \frac{1}{k^2}\, \langle \mathbf{v}, \mathbf{v}_k \rangle$ is convergent if $\mathbf{v} \in \mathcal{H}$;

(ii) That $\Phi$ is linear;

(iii) That $\Phi$ is bounded.

In order to prove (i), we will show that $\sum_{k=1}^{\infty} \frac{1}{k^2}\, \langle \mathbf{v}, \mathbf{v}_k \rangle$ is absolutely convergent using Cauchy–Schwarz' inequality:

$$\begin{aligned} \sum_{k=1}^{\infty} \left| \frac{1}{k^2}\, \langle \mathbf{v}, \mathbf{v}_k \rangle \right| &\leq \sum_{k=1}^{\infty} \frac{1}{k^2}\, ||\mathbf{v}||\, ||\mathbf{v}_k|| \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2}\, ||\mathbf{v}|| \\ &< \infty. \end{aligned} \tag{12.6}$$

Since an absolutely convergent series is convergent, we have proved (i).

(ii) follows by direct verification: for any $\mathbf{v}, \mathbf{w} \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} \Phi(\alpha\mathbf{v} + \beta\mathbf{w}) &= \sum_{k=1}^{\infty} \frac{1}{k^2}\, \langle \alpha\mathbf{v} + \beta\mathbf{w}, \mathbf{v}_k \rangle \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2}\, \left(\alpha\langle \mathbf{v}, \mathbf{v}_k \rangle + \beta\langle \mathbf{w}, \mathbf{v}_k \rangle\right) \\ &= \alpha \sum_{k=1}^{\infty} \frac{1}{k^2} \langle \mathbf{v}, \mathbf{v}_k \rangle + \beta \sum_{k=1}^{\infty} \frac{1}{k^2} \langle \mathbf{w}, \mathbf{v}_k \rangle \\ &= \alpha\Phi\mathbf{v} + \beta\Phi\mathbf{w}. \end{aligned}$$

In order to prove (iii), we use the triangle inequality and the calculation in (i), see (12.6):

$$\begin{aligned}|\Phi \mathbf{v}| &= \left|\sum_{k=1}^{\infty} \frac{1}{k^2} \langle \mathbf{v}, \mathbf{v}_k \rangle \right| \\ &\le \sum_{k=1}^{\infty} \left| \frac{1}{k^2} \langle \mathbf{v}, \mathbf{v}_k \rangle \right| \\ &\le \sum_{k=1}^{\infty} \frac{1}{k^2} ||\mathbf{v}||.\end{aligned}$$

This proves that $\Phi$ is bounded, with

$$||\Phi|| \le \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Note that, according to Riesz' representation theorem, $\Phi$ has the form

$$\Phi \mathbf{v} = \langle \mathbf{v}, \mathbf{w} \rangle$$

for some $\mathbf{w} \in \mathcal{H}$. Let us calculate $\mathbf{w}$ :

$$\begin{aligned}\Phi \mathbf{v} &= \sum_{k=1}^{\infty} \frac{1}{k^2} \langle \mathbf{v}, \mathbf{v}_k \rangle \\ &= \sum_{k=1}^{\infty} \langle \mathbf{v}, \frac{1}{k^2} \mathbf{v}_k \rangle \qquad (12.7) \\ &= \langle \mathbf{v}, \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{v}_k \rangle. \qquad (12.8)\end{aligned}$$

Note that the step from (12.7) to (12.8) uses the result in Exercise 4.25. Thus,

$$\mathbf{w} = \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{v}_k.$$

□

# Quantum Mechanics

**Classical Physics:** The position (Danish: "sted") and momentum (Danish: "impuls") of a physical object can in principle be determined with arbitrary precision simultaneously.

**Works well for large-scale objects!**

**Quantum Mechanics for small-scale objects like particles:** A measurement of position and momentum affects the physical object, so exact values of position and momentum can not be given.

(*Danish: En måling af f.eks. sted eller impuls påvirker det der måles på, så en eksakt fastlæggelse af sted og impuls er umulig*)

We can only predict that a particle will have a certain position or momentum in a probabilistic sense.

(*Danish: Sted og impuls kan kun fastlægges i en sandsynlighedsteoretisk forstand, altså ud fra en bestemt sandsynlighedsfordeling. Med andre ord - vi kan kun angive en sandsynlighed for at partiklen til et givet tidspunkt vil befinde sig et givet sted.*)

From now on - consider a "physical system."

**Physical system:** Think about a particle that moves around in a potential.

**Quantum Mechanics:**

- Consists of a series of postulates (axioms)
- The postulates are justified by experimental verification of their consequences.

**Postulate 1:** To a physical system we can associate a *wave function* (Danish: "bølgefunktion") that contains all the relevant information about the system.

Consider a physical system consisting of a single particle. Let

- $\mathbf{r}$ denote the position of the particle;
- $t$ denote the time.

The wave function has the form $\Psi(\mathbf{r}, t)$ and

$$\int_{\mathbb{R}^3} |\Psi(\mathbf{r}, t)|^2 \, d\mathbf{r} < \infty, \quad \forall\, t \in \mathbb{R}.$$

If this is independent of $t$ : by proper normalisation,

$$\int_{\mathbb{R}^3} |\Psi(\mathbf{r}, t)|^2 \, d\mathbf{r} = 1, \ \forall\, t \in \mathbb{R}.$$

**Interpretation:** $|\Psi(\mathbf{r}, t)|^2$ is a probability density, i.e., $|\Psi(\mathbf{r}, t)|^2 \, d\mathbf{r}$ measures the probability of finding the particle in the volume element $d\mathbf{r}$ around $\mathbf{r}$.

**Postulate 3:** To every observable (for example position, momentum, energy) we can associate a linear operator

$$A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3).$$

For example, the position of a particle moving in a potential $V(\mathbf{r}, t)$ is described by the *Schrödinger operator*

$$A\Psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + V(\mathbf{r}, t) \right] \ \Psi(\mathbf{r}, t)$$
$$\left( = i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) \right),$$

where $\hbar$ os a physical constant.

**Postulate 4:** The only possible values of the observables are eigenvalues for the associated operator $A$, i.e., the numbers $\lambda_n$ for which there exist functions $\psi_n \neq 0$ such that

$$A\psi_n = \lambda_n \psi_n.$$

**Conclusion:** Quantum mechanics deals with the Hilbert space $L^2(\mathbb{R}^3)$ and the operators on that space. Other Hilbert spaces appear if we consider collections of particles.

**We cite Roger Penrose, who said:**

Quantum theory has two things in its favour and only one against.

First, it agrees with all the experiments

Second, it is a theory of astonishing and profound mathematical beauty

The only thing to be said against the theory is that it makes absolutely no sense.