12.2 Week 2

12.2.1 Selected results

2.10: (i) yes; (ii) yes; (iii) no.
3.14: (iv) T is an isometry, S is not; (v) T is injective, S is not; (vi) T is not surjective, but S is surjective; $TS\mathbf{x} = (0, x_2, x_3, \dots)$

12.2.2 Selected solutions

Exercise 2.14:(i) Since

$$||T\mathbf{w} - T\mathbf{w}_k|| = ||T(\mathbf{w} - \mathbf{w}_k)|| \leq ||T||\,||\mathbf{w} - \mathbf{w}_k|| \to 0 \text{ as } k \to \infty,$$

we see that

$$T\mathbf{w}_k \to T\mathbf{w} \text{ as } k \to \infty.$$

(ii) Since

$$\begin{aligned} \left|\left|T\sum_{k=1}^{\infty} c_k\mathbf{v}_k - \sum_{k=1}^{N} c_kT\mathbf{v}_k\right|\right| &= \left|\left|T\left(\sum_{k=1}^{\infty} c_k\mathbf{v}_k - \sum_{k=1}^{N} c_k\mathbf{v}_k\right)\right|\right| \\ &\leq ||T||\left|\left|\sum_{k=1}^{\infty} c_k\mathbf{v}_k - \sum_{k=1}^{N} c_k\mathbf{v}_k\right|\right| \\ &\to 0 \text{ as } N \to \infty, \end{aligned}$$

we see that

$$\sum_{k=1}^{N} c_kT\mathbf{v}_k \to T\sum_{k=1}^{\infty} c_k\mathbf{v}_k \text{ as } N \to \infty.$$

By definition, this means that

$$\sum_{k=1}^{\infty} c_kT\mathbf{v}_k = T\sum_{k=1}^{\infty} c_k\mathbf{v}_k.$$

Exercise 3.1: Assume that $\{\mathbf{v}_k\}_{k=1}^{\infty}$ is convergent, $\mathbf{v}_k \to \mathbf{v}$ as $k \to \infty$. We shall show that there for each $\epsilon > 0$ exists an $N \in \mathbb{N}$ such that

$$||\mathbf{v}_k - \mathbf{v}_\ell|| \leq \epsilon \text{ for } k, \ell \geq N.$$

Now, according to the triangle inequality,

$$||\mathbf{v}_k - \mathbf{v}_\ell|| = ||\mathbf{v}_k - \mathbf{v} + \mathbf{v} - \mathbf{v}_\ell|| \leq ||\mathbf{v}_k - \mathbf{v}|| + ||\mathbf{v} - \mathbf{v}_\ell||. \tag{12.3}$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that

$$||\mathbf{v} - \mathbf{v}_k|| \leq \epsilon/2 \text{ for } k \geq N;$$

then, for $k, \ell \geq N$, the inequality (12.3) shows that

$$||\mathbf{v}_k - \mathbf{v}_\ell|| \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

as desired.

Exercise 3.14: (i) Consider S and T as matrices. If $ST = I$, then

$$\det(S)\ \det(T) = \det(ST) = 1;$$

this implies that $\det(S) \neq 0$, and therefore S is invertible. Multiplying $ST = I$ with S^{-1} from the left now implies that $S^{-1}ST = S^{-1}$, i.e., that $T = S^{-1}$; multiplying this equation with S from the right implies that $TS = I$.

The same argument shows that if $TS = I$, then it also holds that $ST = I$.
(ii) The operators T and S are linear (check that!). Let $\mathbf{x} = (x_1, x_2, x_3, \dots)$.. Then

$$\begin{aligned} ||T\mathbf{x}|| &= ||(0, x_1, x_2, x_3, \dots)|| \\ &= 0 + \sum_{k=1}^{\infty} |x_k| \\ &= ||\mathbf{x}||. \end{aligned}$$

This shows that T is bounded and that $||T|| = 1$. Similarly,

$$\begin{aligned} ||S\mathbf{x}|| &= ||(x_2, x_3, \dots)|| \\ &= \sum_{k=2}^{\infty} |x_k| \\ &\leq \sum_{k=1}^{\infty} |x_k| \\ &= ||\mathbf{x}||. \end{aligned}$$

This proves that S is bounded.
(iii) We only need the argument for S. By the calculation in (ii) we know that $||S|| \leq 1$. Taking $\mathbf{x} = (0, 1, 0, 0 \dots)$, we have

$$||S\mathbf{x}|| = ||(1, 0, 0, \dots)|| = 1 = ||\mathbf{x}||.$$

This proves that actually $||S|| = 1$.
(iv) The calculation in (ii) shows that T is an isometry. Letting $\mathbf{x} := (1, 0, 0, \dots)$, we see that

$$||\mathbf{x}|| = 1, \quad ||S\mathbf{x}|| = ||\mathbf{0}|| = 0.$$

Thus S is not an isometry.
(v) T is injective because

$$T\mathbf{x} = \mathbf{0} \Rightarrow (0, x_1, x_2, \dots) = 0 \Rightarrow x_k = 0,\ \forall k.$$

S is not injective: we have seen that

$$S(1,0,0,\dots) = \mathbf{0}.$$

(vi) T is not surjective - for example the vector $(1,0,0,\dots)$ does not belong to the range of T. The operator S is surjective: given any $\{y_k\} \in \ell^1(\mathbb{N})$, we have

$$\{y_k\} = S(0, y_1, y_2, \dots).$$

(vii) For any $\mathbf{x} \in \ell^1(\mathbb{N})$,

$$ST\mathbf{x} = S(0, x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots) = \mathbf{x},$$

but

$$TS\mathbf{x} = T(x_2, x_3, \dots) = (0, x_2, x_3, \dots).$$

That is, if $x_1 \neq 0$, then

$$TS\mathbf{x} \neq ST\mathbf{x}.$$

This proves that $TS \neq ST$.

Exercise 2.13: (i) We must show that if $\mathbf{v} \in \overline{W}$, then $T\mathbf{v} \in \overline{T(W)}$. Now, for $\mathbf{v} \in \overline{W}$, we can by definition find a sequence of elements $\{\mathbf{v}_k\}_{k=1}^{\infty}$ in W such that

$$\mathbf{v}_k \to \mathbf{v} \text{ as } k \to \infty.$$

Since

$$||T\mathbf{v} - T\mathbf{v}_k|| = ||T(\mathbf{v} - \mathbf{v}_k)|| \leq ||T||\,||\mathbf{v} - \mathbf{v}_k|| \to 0 \text{ as } k \to \infty,$$

this implies that

$$T\mathbf{v}_k \to T\mathbf{v} \text{ as } k \to \infty.$$

Since $T\mathbf{v}_k \in T(W)$, this means by definition that $T\mathbf{v} \in \overline{T(W)}$.
(ii) If we apply the result in (i) with T replaced by the operator T^{-1} and the set W replaced by TW, we arrive at

$$T^{-1}(\overline{TW}) \subseteq \overline{T^{-1}TW} = \overline{W}.$$

Applying the operator T on this yields that $\overline{TW} \subseteq T(\overline{W})$. Combined with the result in (i) this concludes the proof.

12.2.3 Examples from the lecture

Example 12.2.1 Consider the mapping

$$T : C(0,2) \to C(0,2),\ (Tf)(x) := x^2 f(x),\ x \in [0,2].$$

We want to show the following:

(i) That T actually maps $C(0,2)$ into $C(0,2)$, i.e., that $Tf \in C(0,2)$ if $f \in C(0,2)$;

(ii) That T is linear;

(iii) That T is bounded.

To prove (i), note that if f is a continuous function on $[0,2]$, then also $x^2 f(x)$ is a continuous function on $[0,2]$; thus, $Tf \in C(0,2)$. To prove that T is linear, let $f, g \in C(0,2)$ and $\alpha, \beta \in \mathbb{C}$. Then, for any $x \in [0,2]$,

$$\begin{aligned}(T(\alpha f + \beta g))(x) = x^2(\alpha f(x) + \beta g(x)) &= x^2 \alpha f(x) + x^2 \beta g(x) \\ &= \alpha (Tf)(x) + \beta (Tg)(x).\end{aligned}$$

This proves that

$$T(\alpha f + \beta g) = \alpha Tf + \beta Tg.$$

For (iii), let again $f \in C(0,2)$, and let $x \in [0,2]$. Then

$$\begin{aligned}|(Tf)(x)| = |x^2 f(x)| &\leq \max_{x\in[0,2]} x^2 \max_{x\in[0,2]} |f(x)| \\ &= 4\,||f||_\infty.\end{aligned}$$

Since this holds for all $x \in [0,2]$, we conclude that

$$||Tf||_\infty = \max_{x\in[0,2]} |(Tf)(x)| \leq 4\,||f||_\infty.$$

This shows that T is bounded, and that $||T|| \leq 4$. □

Example 12.2.2 Consider the mapping

$$T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}),\ T\{x_k\}_{k=1}^\infty := \{x_k + x_{k+1}\}_{k=1}^\infty.$$

We want to show the following:

(i) That T actually maps $\ell^p(\mathbb{N})$ into $\ell^p(\mathbb{N})$, i.e., that $T\{x_k\}_{k=1}^\infty \in \ell^p(\mathbb{N})$ if $\{x_k\}_{k=1}^\infty \in \ell^p(\mathbb{N})$;

(ii) That T is linear;

(iii) That T is bounded.

In order to prove (i), we use Minkovski's inequality, Theorem 1.7.3:

$$\begin{aligned} ||T\{x_k\}_{k=1}^{\infty}||_{\ell^p(\mathbb{N})} &= \left(\sum_{k=1}^{\infty} |x_k + x_{k+1}|^p\right)^{1/p} \\ &\leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |x_{k+1}|^p\right)^{1/p} \\ &= \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=2}^{\infty} |x_k|^p\right)^{1/p} \\ &\leq 2\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \\ &< \infty. \end{aligned}$$

This proves (i). (ii) follows by direct verification. In order to prove (iii), we continue the calculation in (i):

$$\begin{aligned} ||T\{x_k\}_{k=1}^{\infty}||_{\ell^p(\mathbb{N})} &\leq 2\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \\ &= 2\,||\{x_k\}_{k=1}^{\infty}||_{\ell^p(\mathbb{N})}. \end{aligned}$$

This proves that T is bounded, with $||T|| \leq 2$. □