12

Weekly Notes

12.1 Week 1

12.1.1 Selected results

1.8: (i) 1, also a maximum; (ii) 1, also a maximum; (iii) 1, not a maximum; (iv) 1, not a maximum.
2.3: (i) Yes, $\overline{\mathbb{N}} = \mathbb{N}$; (ii) No, $\overline{\mathbb{Q}} = \mathbb{R}$. (see solution)

12.1.2 Selected solutions

Exercise 1.4 - Proof of Lemma 1.2.7: First assume that (1.11) holds. Then the set W satisfies the conditions $\mathbf{v}, \mathbf{w} \in W \Rightarrow \mathbf{v} + \mathbf{w} \in W$ and $\mathbf{v} \in W, \lambda \in \mathbb{C} \Rightarrow \lambda\mathbf{v} \in W$. The set W also satisfies the conditions (i)-(ii) and (v)-(viii) in Definition 1.2.1 because they hold on V and $W \subseteq V$. Take any $\mathbf{v} \in W$ - then the assumption (1.11) shows that $\mathbf{0} = 0\mathbf{v} \in W$, i.e., (iii) holds. (iv) is proved similarly by multiplying $\mathbf{v} \in W$ by the scalar -1. Thus W is a subspace of V.

On the other hand, if W is a subspace of V we know that the sum of two elements in W belongs to W, and that a scalar multiple of a vector in W belongs to W. Combining these two facts, we see that (1.11) holds.

Exercise 2.1: Assume that $\mathbf{v}_k \to \mathbf{v}$ as $k \to \infty$; by definition, this means that

$$||\mathbf{v} - \mathbf{v}_k|| \to 0 \text{ as } k \to \infty. \tag{12.1}$$

According to Lemma 2.1.2,

$$| \, ||\mathbf{v}|| - ||\mathbf{v}_k|| \, | \le ||\mathbf{v} - \mathbf{v}_k||;$$

due to (12.1) this implies that

$$\lim_{k\to\infty} | \, ||\mathbf{v}|| - ||\mathbf{v}_k|| \, | = 0.$$

Since $||\mathbf{v}||$ is a constant, it follows that

$$\lim_{k\to\infty} ||\mathbf{v}_k|| = ||\mathbf{v}||,$$

as desired.

Exercise 2.3: (i) $\mathbb{N}$ is closed because $\mathbb{R} \setminus \mathbb{N}$ is open (make a figure!). So $\overline{\mathbb{N}} = \mathbb{N}$.

(ii) For any $x \in \mathbb{R}$ there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \to x$ as $k \to \infty$. This proves that $\overline{\mathbb{Q}} = \mathbb{R}$, and also shows that $\mathbb{Q}$ is non-closed.

Exercise 2.4: (i) The trigonometric polynomials are continuous, so they belong to $C(0,1)$. A linear combination of trigonometric polynomials is again a trigonometric polynomial, so Lemma 1.2.7 implies that V is a subspace of $C(0,1)$.

In order to examine the question in (ii), consider the function

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{k^2} e^{2\pi i k x}. \tag{12.2}$$

The infinite series in (12.2) has a convergent majorant series, so the function f is continuous (see Chapter 5 in the MAT2-book). However, by definition, the function f is *not* a trigonometric polynomial.

Now consider the partial sums

$$f_N(x) := \sum_{k=1}^{N} \frac{1}{k^2} e^{2\pi i k x}, \ N \in \mathbb{N}.$$

Each of the functions f_N is a trigonometric polynomial. We also note that for all $x \in [0,1]$,

$$\begin{aligned} |f(x) - f_N(x)| &= \left| \sum_{k=N+1}^{\infty} \frac{1}{k^2} e^{2\pi i k x} \right| \\ &\le \sum_{k=N+1}^{\infty} \left| \frac{1}{k^2} e^{2\pi i k x} \right| \\ &= \sum_{k=N+1}^{\infty} \frac{1}{k^2}. \end{aligned}$$

This implies that

$$\begin{aligned} ||f - f_N||_\infty &= \sup_{x\in[0,1]} |f(x) - f_N(x)| \\ &\leq \sum_{k=N+1}^{\infty} \frac{1}{k^2} \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

That is, we have found a sequence of functions in V that converges to a function that is not in V. Thus V is non-closed.

12.1.3 Slides

Mathematics 2:

- Understand advanced math.
- Apply advanced math.

Mathematics 4 (Math. 2H, 3):

- Understand advanced math.
- Apply advanced math.
- **Prove** advanced math.
- **Derive** advanced math.

The central point is not to “calculate something”.

The central point is to understand the concepts and how they are related.

Math. 3 versus Math. 4:

- The mathematical goals are different;
- Some overlap in the basic tools, treated the first 4 weeks.

Mathematics 4:

- A *toolbox,* presenting topics that play key roles in mathematics, physics, engineering;
- Entrance to advanced mathematical analysis.

Central themes in Math. 4:

1) Abstract concepts:

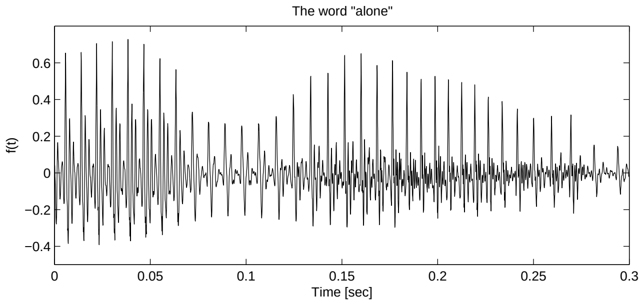
- Normed vector spaces
- Hilbert spaces
- Bases in Hilbert spaces
- Basic operator theory

2) Concrete vector spaces:

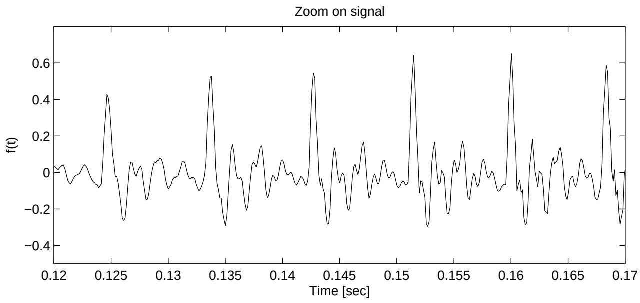
- The spaces L^p and ℓ^p
- The spaces L^2 and ℓ^2

3) Concrete functions:

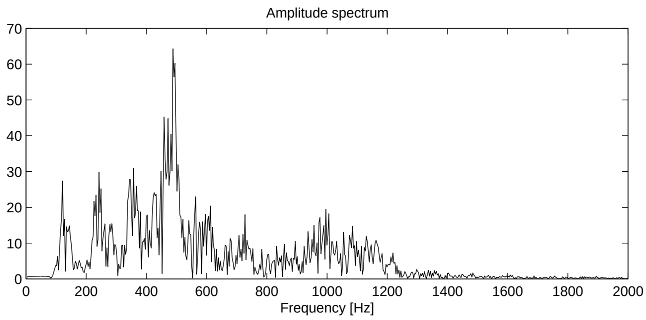
- The Fourier transform
- Wavelet theory
- B-Splines
- Basic Sturm-Liouville theory
- Legendre and Hermite polynomials



A speech signal. One can think about such a signal as the current in the cable to the loudspeaker when a recording of the speech is played. The actual signal is a recording of the word "alone".



A part of the signal.



The Fourier transform of the signal, showing the frequencies appearing in the signal.