01234 Differential Geometry with Applications

Project Exercise: Cut Locus Geometry

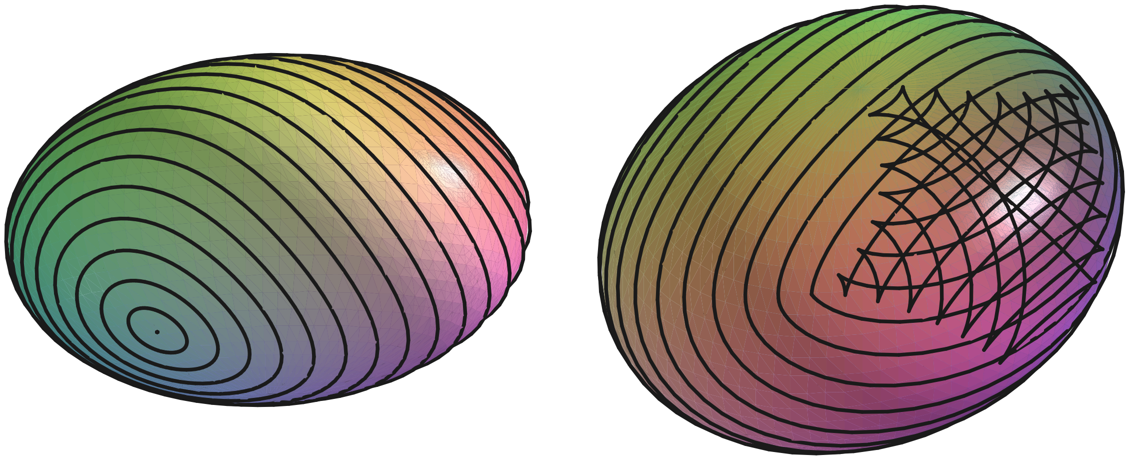

Figure 1: An ellipsoid with concentric geodesic circles. The center point of the circles is visible to the left. The cut locus of the center point is created on 'the other side' of the ellipsoid and can be extracted from the criss-cross of the self-intersecting geodesic circles.

1 The problem

When walking on a given surface along a geodesic (away) from a point p you have passed a *cut point* of p if there is another shorter curve back to p from the point you have reached. For example, along every geodesic starting from the north pole of a sphere the south pole is a cut point - and vice versa.

The cut locus $\mathcal{C}(p)$ of a point p on a surface F is by definition the set of cut points of p. In rough terms the cut locus consists of those points in the surface where the geodesic circle from p 'crosses over itself' when progressing through increasing radii as indicated in figure 1.

Cut loci have applications in e.g. robotics: to understand globally what are the shortest pathways in configuration space; and in optics: to understand the structure of focal points of light rays radiating from a point and constrained to lie in a surface.

On a sphere we have $\mathcal{C}(NorthPole) = SouthPole$ and $\mathcal{C}(SouthPole) = NorthPole$. But what does the cut locus look like on more general surfaces? What is the structure of the cut locus for various points on, say, a cylinder, an ellipsoid, a paraboloid, a torus, a surface of revolution?

There is an interesting conjecture - and several nice pictures - in the recent paper [3] by R. Sinclair and M. Tanaka concerning the structure of cut loci on surfaces of revolution of torus type generated by closed profile curves in the profile plane.

2 Methods involved

Solving the geodesic differential equation system together with a method to trace the crossings of the geodesic circles from given initial points and a method to extract the cut locus from these crossings.

2.1 Estimated specific types of work loads

- Theory: ***
- Maple: ****

References

[1] J.Gravesen, S. Markvorsen, R. Sinclair and M. Tanaka, *The Cut Locus of a Torus of Revolution*, Asian Journal of Mathematics **9** No. 1 (2005) 103–120.

[2] A. Pressley, *Elementary Differential Geometry*, Springer, 2010.

[3] R. Sinclair and M. Tanaka, *A Bound on the Number of Endpoints of the Cut Locus*, London Mathematical Society, J. Comput. Math. **9** (2006) 21–39.

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