Mathematical problems for Moineau pumps

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1 The problem

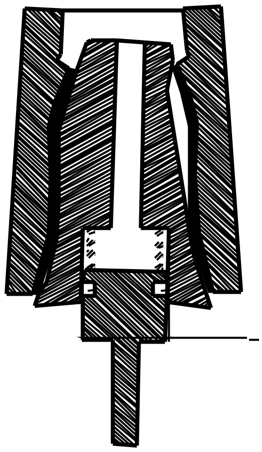
A progressive cavity pump (PCP) or a Moineau pump has two parts rotating relative to each other and moving in an eccentric track relative to each other. The shapes in an axial cross section can be made of pieces of hypocycloids connected with epicycloids. Alternative it can be a curve with a constant distance to hypocycloids.

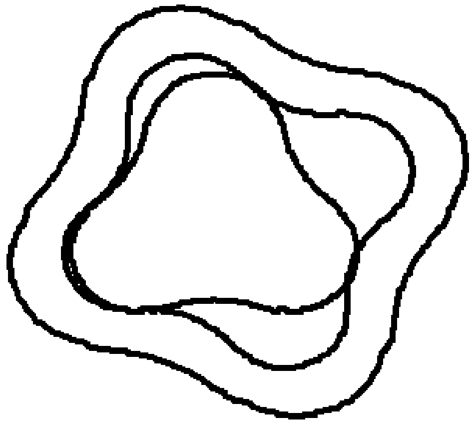
In the axial direction the cross section is rotated around the z-axis so the external part have at least one thread.

For n-teeth shape running inside an n+1 teeth shape the internal part have (n+1)/n times the teeth of the external part.

The centre of the internal element is offset relative to the centre of the external element and moving in a circle with the radius e (eccentricity). The speed of displacement of the centres is -n times the speed of the rotation speed of the internal element if the external element is stationary.

A further feature can be achieved by making the pump conical. If both elements have a larger cross section in one end the axial force will determine the compression between the two elements. In that configuration one of the elements can be free to move in axial direction and the pump pressure act on a surface can presses the two elements together. By adjusting the area of the surface where the pressure compresses the two parts the closing force for the pump can be adjusted. If the pump has to give a constant flow the cross section of the cavities has to be constant (if thread height is constant) in the z direction. Keeping the cross section of the cavities constant (area between external surface and internal surface) can be obtained by decreasing the eccentricity towards the large end of the pump. Eccentricity has to be increased linear if the parts are rigid.





- For n-teeth running inside n+1 teeth: make an expression for as well internal as well as external element with different eccentricities.
- If the pump were conical in z-direction what would be the expression for the elements if the cross section of the cavities and thread height thread height should be constant for a linear increasing eccentricity in the z direction.
- Expression describing the flow for the pump above (Length L, Speed v (Hz))
- The pump making a pressure P: What would be the axial force on inner or external element as a function of the turning angle?
- What would be the radial forces as a function on turning angle?

2 Introduction and notation

We have solely considered the geometry of pump and have not made any effort to calculate forces or mechanical properties of the different designs.

We first consider motions where the axis of rotation has a fixed direction, i.e., it moves on a general cylinder. The motion then preserves planes orthogonal to the axis of rotation and is completely determined by the motion in one of these planes. Such a planar motion is given by a pair of curves, called *pole curves*, rolling on each other. In Theorem 1 we prove that the pole curves has to be circles and hence that the axis of rotation sweeps out a circular cylinder.

For the other motion the axis of rotation goes through a fixed point, i.e., it moves on a general cone. The motion now preserves spheres and is completely determined by the motion on one of these spheres. We are convinced that the planar case generalises to this case so the motion is given by a pair of circles on a sphere that rolls on each other. Thus the axis of rotation sweeps out a circular cone.

We are now looking for a pair of curves, giving the inner and outer part of the pump. They should be designed such that they during the motion do not intersect but touches in a number of points and in this way gives a number of pump chambers. Furthermore, during the motion the chambers has to disappear at certain times, i.e., the points of contact has to coincide at certain times.

We first investigate the classical design consisting of hypo- or epicycloids, see Figure 2 and 6. We show that they do work, i.e., they behave as described above, and we calculate the area of the pump chambers as a function of time, see Theorem 2 and 3 and Figure 3 and 7. We also show that it is *impossible* to offset the cycloids and thereby obtain a design without cusps, see Figure 4 and 5. Impossible means that the design does not work mathematically exact, in practice the rubber sealing might make it work.

We then investigate a design consisting of alternating arcs of hypo and epi cycloids, see Figure 8. This design works too, and we give expressions for the the area of the pump chambers as a function of time, see Theorem 7 and Figure 10. Even though the design is without cusps, there are points with infinite curvature, so it is once more *impossible* to offset the design.

In the next design the inner part is given as a general hypotrochoid, see Figure 11. We determine the outer part. Graphically it works, see Figure 14, but we have not proved it and we can no longer find an analytical expression for the area of the pump chambers. It is of course possible to do it numerically. This design might have continuous and bounded curvature and then it would be possible to offset it and get a new design, but we have not pursued that aspect.

In the last two designs mentioned the inner part is made by two segments one with positive curvature and one with negative curvature joined at an inflexion point. These two segments span the angle $2\pi/n$, the whole inner part is now obtained by taking n copies rotated through the angle $2\ell\pi/n$ for $\ell = 0, \dots, n-1$. In Section 5 we represent the positively curved segment by its support function. The whole outer part can now be found as the envelope of this single segment. By interchanging the role of the fixed and moving part the whole inner part can now be found as the envelope of the outer part. It is also possible to find the area of the inner and outer part of the design in terms of the support function, see (112) and (113).

We finally consider spherical designs in Section 6. We analyse the spherical analog of the hypo- and epi-cycloid design, but it turns out that it does not work. By construction there is one point of contact, but the n cusps of the inner hypocycloid do no longer touch the outer hypocycloid. As in the case of offsets of the planar design, the error is probably so small that the rubber sealing can take care of it.

Before we start the analysis we define two unit vectors in the plane and a rotation matrix:

$$\mathbf{e}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}, \qquad \mathbf{R}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}. \tag{1}$$

The vectors $\mathbf{e}(t), \mathbf{f}(t)$ form an orthonormal basis for each $t \in \mathbb{R}$. We furthermore have the following useful equations

$$\mathbf{R}(t)\mathbf{e}(\phi) = \mathbf{e}(\phi + t), \tag{2}$$
$$\mathbf{R}(t)\mathbf{f}(\phi) = \mathbf{f}(\phi + t), \tag{3}$$
$$\mathbf{e}(\phi) \cdot \mathbf{e}(\theta) = \mathbf{e}(\phi - \theta) \cdot \mathbf{e}(0) = \cos(\phi - \theta), \tag{4}$$
$$\mathbf{e}(\phi) \cdot \mathbf{f}(\theta) = \mathbf{e}(\phi - \theta) \cdot \mathbf{f}(0) = \sin(\phi - \theta), \tag{5}$$
$$\mathbf{f}(\phi) \cdot \mathbf{e}(\theta) = \mathbf{f}(\phi - \theta) \cdot \mathbf{e}(0) = -\sin(\phi - \theta), \tag{6}$$
$$\mathbf{f}(\phi) \cdot \mathbf{f}(\theta) = \mathbf{f}(\phi - \theta) \cdot \mathbf{f}(0) = \cos(\phi - \theta), \tag{7}$$
$$\mathbf{e}(\theta + \phi) = \cos(\phi)\mathbf{e}(\theta) + \sin(\phi)\mathbf{f}(\theta). \tag{8}$$
$$\mathbf{f}(\theta + \phi) = -\sin(\phi)\mathbf{e}(\theta) + \cos(\phi)\mathbf{f}(\theta). \tag{9}$$

3 The possible cylindrical motion

We consider two different types of motions. In both cases the velocity field is from a pure rotation, so we exclude rest, translation and screw motion. In the first case the instantaneous axis of rotation moves on a cylinder and in the second case the instantaneous axis of rotation moves on a cone. We call the two cases cylindrical motion and conical motion respectively. We shall later see that the cylinder has to be circular and we are convinced that the same proof also applies in the conical case, so the cone has to be circular too.

When the instantaneous axes of rotation moves on a cylinder or equivalently are orthogonal to a fixed plane the motion is essentially planar. Furthermore, as the now planar motion we are considering always has a rotational component, it can be realised as the rolling of one curve upon another.

We can think of the inner or outer part of the pump as planar sections (orthogonal to the axis of rotation) stacked on top of each other. The design is such that each planar section is a fixed curve just rotated relative to each other. Not only are the two parts made this way but if we look at the motion of one part relative to the other in one planar section, then we see the same “movie” in all the sectioons. The only difference is that movies are rotated relative to each other and that they have a different time offset in the different sections. Consider the situation in Figure 1.

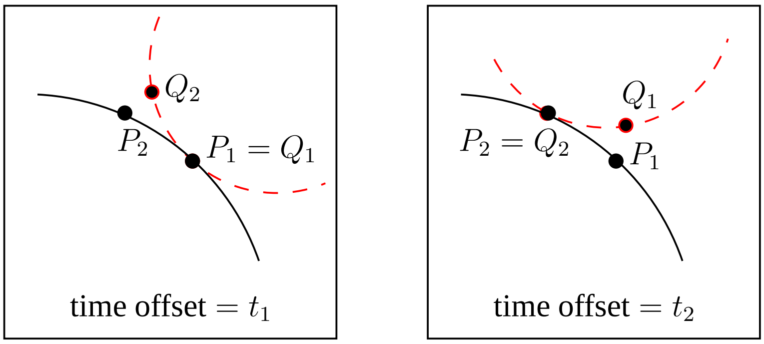


Figure 1: The motion at different time offsets

We have two pictures at different times or rather two movies with different time offsets. We now want to put these two movies on top of each other and as the inner and outer parts are rigid objects it has to be done in such a way that the two moving planes (red stippled) moves together. That means that the point $P_1 = Q_1$ at time t_1 has to be on top the point $P_2 = Q_2$ at time t_2 and that the fixed pole curve (black solid) and the moving pole curve (red stippled) are put on top of each other. But then the curvature at P_1 and P_2 on the fixed pole curve is the same. Similar, the curvature at Q_1 and Q_2 on the moving pole curve is the same. As the time offsets t_1 and t_2 was arbitrary we can conclude that both the fixed and the moving pole curve has constant curvature. But then they are circles and we have proved the following result:

Theorem 1. *The only possible cylindrical motion is given by circles rolling on circles, or equivalently it is given by*

$$\mathbf{x} \mapsto \mathbf{c} + c\mathbf{e}(\alpha t + \beta) + \mathbf{R}(t)\mathbf{x}.$$

The fixed pole curve F *is a circle with centre* $\mathbf{c}$ *and radius* $c(1-\alpha)$ *and the rolling pole curve* R *is a circle with centre* $\mathbf{0}$ *and radius* $c\alpha$*:*

$$\mathsf{F}(t) = \mathbf{c} + c(1-\alpha)\mathbf{e}(\alpha t + \beta), \quad \textit{and} \quad \mathsf{R}(t) = -c\alpha\mathbf{e}(\alpha t + \beta).$$

Proof. We only need to determine the pole curves. The velocity field is

$$\mathbf{v}(\mathbf{x}) = c\alpha\mathbf{f}(\alpha t + \beta) + \mathbf{R}'(t)\mathbf{x} = c\alpha\mathbf{f}(\alpha t + \beta) + \mathbf{R}\left(t + \frac{\pi}{2}\right)\mathbf{x}.$$

and we see that $\mathbf{v} = \mathbf{0}$ if and only if

$$\mathbf{x} = \mathbf{R}\left(-t - \frac{\pi}{2}\right)(-c\alpha\mathbf{f}(\alpha t + \beta)) = -c\alpha\mathbf{e}((\alpha - 1)t + \beta),$$

which is the rolling pole curve. The fixed pole curve is now found by finding the corresponding point in the fixed plane, i.e.,

$$\mathbf{y} = \mathbf{c} + c\mathbf{e}(\alpha t + \beta) + \mathbf{R}(t)\mathbf{x} = \mathbf{c} + c(1-\alpha)\mathbf{e}(\alpha t + \beta). \qquad \square$$

4 Geometric pump constructions

Starting with the original construction by Moineau using hypocycloids and epicycloids we give some geometric constructions of possible pump designs.

4.1 The classical hypocycloids and epicycloids

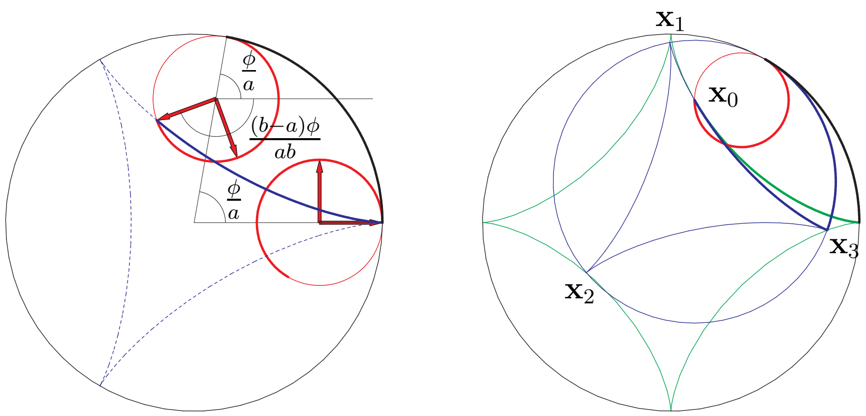


Figure 2: To the left the parametrisation of a hypocycloid. To the right the left picture is rolled inside a circle with radius $n+1$ such that the three thick circle segments all have length t.

Theorem 2. *The hypocycloid obtained by rolling a circle of radius 1 inside a circle of radius n can be parametrised as*

$$\mathbf{x}_{h,n}(\phi) = (n-1)\mathbf{e}\left(\frac{\phi}{n}\right) + \mathbf{e}\left(\frac{(1-n)\phi}{n}\right), \quad t \in [0, 2n\pi], \tag{10}$$

where ϕ is the distance the small circle has rolled, see Figure 2.

If it is moved by the motion obtained by rolling a circle of radius n inside a circle of radius $n+1$ in such a way that the point of contact between the two circles has speed 1, then at time t the point $\mathbf{x}_{h,n}(t)$ touches the hypocycloid $\mathbf{x}_{h,n+1}$ in the point $\mathbf{x}_{h,n+1}(t)$. Furthermore, the n cusps on the moving hypocycloid $\mathbf{x}_{h,n}$ have contact with the fixed hypocycloid $\mathbf{x}_{h,n+1}$ at the points

$$\mathbf{x}_{h,n+1}\left(\frac{2k(n+1)\pi - t}{n}\right), \quad k = 1 \ldots n. \tag{11}$$

The $n+1$ contact points give $n+1$ chambers which for $t \in [0, 2\pi]$ have areas *given by*

$$A_0 = \frac{(n-1)t}{n} + \frac{\sin t}{n(n+1)} - \frac{n^2}{n+1}\sin\left(\frac{t}{n}\right), \tag{12}$$

$$A_1 = \frac{n-1}{n}(2\pi - t) - \frac{\sin t}{n(n+1)} - \frac{n^2}{n+1}\sin\left(\frac{2\pi - t}{n}\right), \tag{13}$$

and for $k = 2, \ldots, n$

$$A_k = \frac{2(n-1)\pi}{n} - \frac{n^2}{n+1}\left(\sin\left(\frac{2k\pi - t}{n}\right) + \sin\left(\frac{t - 2(k-1)\pi}{n}\right)\right), \tag{14}$$

see Figure 3. The situation at another time $t \in \mathbb{R}$ *is given by symmetry. The total area is*

$$A_{\mathit{total}} = \sum_{k=0}^{n} A_k = 2(n-1)\pi, \tag{15}$$

and the area of the inner part, of the outer part, and of the circumscribed circle is

$$A_{\mathit{inner}} = (n-1)(n-2)\pi, \tag{16}$$

$$A_{\mathit{outer}} = n(n-1)\pi, \tag{17}$$

$$A_{\mathit{circumcircle}} = (n+1)^2\pi. \tag{18}$$

Proof. Consider Figure 2. When a circle with radius b has rolled the distance ϕ inside a circle with radius a the centre is at $(a-b)\mathbf{e}(\phi/a)$, and it has rotated through the angle $(b-a)\phi/(ab)$. So if a point has coordinates (x, y) with respect to a coordinate system with origin in the centre of the moving circle, then it has moved to

$$\mathbf{x}(\phi) = (a-b)\mathbf{e}\left(\frac{\phi}{a}\right) + x\mathbf{e}\left(\frac{(b-a)\phi}{ab}\right) + y\mathbf{f}\left(\frac{(b-a)\phi}{ab}\right). \tag{19}$$

Letting $a = n$, $b = 1$, and $(x, y) = (1, 0)$ shows (10).

The n cusps have coordinates $n\mathbf{e}(2k\pi/n)$, $k = 1 \ldots n$, and moving by letting the circle with radius n roll inside a circle of radius $n+1$, (19) shows that they trace the curves

$$\begin{aligned}\mathbf{x}_k(t) = {} & \mathbf{e}\left(\frac{t}{n+1}\right) \\ & + n\cos\left(\frac{2k\pi}{n}\right)\mathbf{e}\left(\frac{-t}{n(n+1)}\right) + n\sin\left(\frac{2k\pi}{n}\right)\mathbf{f}\left(\frac{-t}{n(n+1)}\right) \\ = {} & \mathbf{e}\left(\frac{t}{n+1}\right) + n\mathbf{e}\left(\frac{2k\pi}{n} - \frac{t}{n(n+1)}\right)\end{aligned}$$

$$= \mathbf{e}\left(\frac{t}{n+1}\right) + n\mathbf{e}\left(\frac{2k(n+1)\pi - t}{n(n+1)}\right)$$

letting $t = 2k(n+1)\pi - n\phi_k$ we get

$$\begin{aligned} &= \mathbf{e}\left(2k\pi - \frac{n\phi_k}{n+1}\right) + n\mathbf{e}\left(\frac{\phi_k}{n+1}\right) = n\mathbf{e}\left(\frac{\phi_k}{n+1}\right) + \mathbf{e}\left(\frac{-n\phi_k}{n+1}\right) \\ &= \mathbf{x}_{h,n+1}(\phi_k) = \mathbf{x}_{h,n+1}\left(\frac{2k(n+1)\pi - t}{n}\right). \end{aligned}$$

Clearly, at time t the point $\mathbf{x}_{h,n}(t)$ on the moving hypocycloid and the point $\mathbf{x}_0(t) = \mathbf{x}_{h,n+1}(t)$ on the fixed hypocycloid coincide. Furthermore, as the line through the instantaneous centre of rotation (the pole) and the point on the traced curve is the normal to the curve, we see that the tangents of the two hypocycloids coincide too. Thus, the hypocycloid $\mathbf{x}_{h,n+1}$ is an envelope.

Alternatively, we can argue as in the case of a general hypotrochoid, see Section 4.3. Indeed, if we let $c = r = 1$ in (52) then we have that the point of contact is determined by the equation

$$\sin\left(\frac{\phi - t}{n}\right) + \sin\left(\phi - \frac{\phi - t}{n}\right) - \sin\phi = 0,$$

or

$$(1 - \cos\phi)\sin\left(\frac{\phi - t}{n}\right) = \left(1 - \cos\left(\frac{\phi - t}{n}\right)\right)\sin\phi.$$

We recognise the solutions $\phi = t + 2nk\pi$ and $\phi = 2k\pi$. Otherwise we have

$$\frac{\sin\phi}{1 - \cos\phi} = \frac{\sin\left(\frac{\phi - t}{n}\right)}{1 - \cos\left(\frac{\phi - t}{n}\right)}$$

and as the derivative of $\frac{\sin\phi}{1-\cos\phi}$ is $-1/(1 - \cos\phi)$ which is strictly negative for $0 < \phi < 2\pi$, the equation is satisfied if and only if

$$\phi = \frac{\phi - t}{n} + 2k\pi \iff \phi = \frac{2kn\pi - t}{n - 1}.$$

This solution gives the inner envelope. So the two hypocycloids $\mathbf{x}_{h,n}$ and $\mathbf{x}_{h,n+1}$ have only the contact points stated in the theorem.

Now e only need to find the area of the chambers during the motion. Due to the symmetry it is enough to investigate the design for $t \in [0, 2\pi]$. For such a t the contact points are on the inner part given by the parameter values t and $2k\pi$, where $k = 1, \ldots, n$. The corresponding points on the outer part are given by the parameter values t and $(2k(n+1)\pi - t)/n$, where $k = 1, \dots, n$. In both cases $k = 0$ gives the same point as $k = n+1$. The parametrisation of the inner part and its derivative is at time t given by

$$\begin{aligned}
\mathbf{x}_t(\phi) &= \mathbf{e}\left(\frac{t}{n+1}\right) + \mathbf{R}\left(\frac{-t}{n(n+1)}\right)\mathbf{x}_{h,n}(\phi), \\
\mathbf{x}'_t(\phi) &= \mathbf{R}\left(\frac{-t}{n(n+1)}\right)\mathbf{x}'_{h,n}(\phi),
\end{aligned}$$

and the planar product, or determinant, of $\mathbf{x}_t$ and $\mathbf{x}'_t$ is

$$\begin{aligned}
&[\mathbf{x}_t(\phi), \mathbf{x}'_t(\phi)] \\
&\quad = \left[\mathbf{e}\left(\frac{t}{n+1}\right) + \mathbf{R}\left(\frac{-t}{n(n+1)}\right)\mathbf{x}_{h,n}(\phi), \mathbf{R}\left(\frac{-t}{n(n+1)}\right)\mathbf{x}'_{h,n}(\phi)\right] \\
&\qquad = \left[\mathbf{R}\left(\frac{t}{n(n+1)}\right)\mathbf{e}\left(\frac{t}{n+1}\right) + \mathbf{x}_{h,n}(\phi), \mathbf{x}'_{h,n}(\phi)\right] \\
&\qquad = \left[\mathbf{e}\left(\frac{t}{n+1} + \frac{t}{n(n+1)}\right) + \mathbf{x}_{h,n}(\phi), \mathbf{x}'_{h,n}(\phi)\right] \\
&= \left[\mathbf{e}\left(\frac{t}{n}\right) + (n-1)\mathbf{e}\left(\frac{\phi}{n}\right) + \mathbf{e}\left(\frac{(1-n)\phi}{n}\right), \right. \\
&\qquad\qquad \left. \frac{n-1}{n}\mathbf{f}\left(\frac{\phi}{n}\right) + \frac{1-n}{n}\mathbf{f}\left(\frac{(1-n)\phi}{n}\right)\right] \\
&= \frac{n-1}{n}\left(\mathbf{f}\left(\frac{t}{n}\right) + (n-1)\mathbf{f}\left(\frac{\phi}{n}\right) + \mathbf{f}\left(\frac{(1-n)\phi}{n}\right)\right) \\
&\qquad\qquad \cdot \left(\mathbf{f}\left(\frac{\phi}{n}\right) - \mathbf{f}\left(\frac{(1-n)\phi}{n}\right)\right) \\
&= \frac{n-1}{n}\left(\cos\left(\frac{t}{n} - \frac{\phi}{n}\right) - \cos\left(\frac{t}{n} - \frac{(1-n)\phi}{n}\right) + (n-1)\right. \\
&\qquad \left. - (n-1)\cos\left(\frac{\phi}{n} - \frac{(1-n)\phi}{n}\right) + \cos\left(\frac{(1-n)\phi}{n} - \frac{\phi}{n}\right) - 1\right) \\
&\quad = \frac{n-1}{n}\left((n-2)(1-\cos\phi) + \cos\left(\frac{\phi - t}{n}\right) - \cos\left(\frac{(n-1)\phi + t}{n}\right)\right).
\end{aligned}$$

The outer part is simply $\mathbf{x}_{h,n+1}(\phi)$ and the planar product with its derivative is

$$\begin{aligned}
&\left[\mathbf{x}_{h,n+1}(\phi), \mathbf{x}'_{h,n+1}(\phi)\right] \\
&= \left[n\mathbf{e}\left(\frac{\phi}{n+1}\right) + \mathbf{e}\left(-\frac{n\phi}{n+1}\right), \frac{n}{n+1}\mathbf{f}\left(\frac{\phi}{n+1}\right) - \frac{n}{n+1}\mathbf{f}\left(-\frac{n\phi}{n+1}\right)\right]
\end{aligned}$$

$$= \frac{n}{n+1}\left(n\mathbf{f}\left(\frac{\phi}{n+1}\right)+\mathbf{f}\left(-\frac{n\phi}{n+1}\right)\right)\cdot\left(\mathbf{f}\left(\frac{\phi}{n+1}\right)-\mathbf{f}\left(-\frac{n\phi}{n+1}\right)\right)$$

$$= \frac{n}{n+1}\left(n-n\cos\left(\frac{\phi}{n+1}+\frac{n\phi}{n+1}\right)+\cos\left(-\frac{n\phi}{n+1}-\frac{\phi}{n+1}\right)-1\right)$$

$$= \frac{n}{n+1}\left(n-n\cos\phi+\cos\phi-1\right)=\frac{n(n-1)}{n+1}(1-\cos\phi).$$

Now

$$\int\left[\mathbf{x}_{h,n+1}(\phi),\mathbf{x}'_{h,n+1}(\phi)\right]\,\mathrm{d}\phi=\frac{n(n-1)}{n+1}\int(1-\cos\phi)\,\mathrm{d}\phi$$

$$=\frac{n(n-1)}{n+1}(\phi-\sin\phi)$$

and

$$\int\left[\mathbf{x}_t(\phi),\mathbf{x}'_t(\phi)\right]\,\mathrm{d}\phi$$

$$=\frac{n-1}{n}\int\left((n-2)(1-\cos\phi)+\cos\left(\frac{\phi-t}{n}\right)-\cos\left(\frac{(n-1)\phi+t}{n}\right)\right)\,\mathrm{d}\phi$$

$$=\frac{(n-1)(n-2)}{n}(\phi-\sin\phi)+(n-1)\sin\left(\frac{\phi-t}{n}\right)-\sin\left(\frac{(n-1)\phi+t}{n}\right).$$

Using this and Maple we find that the area of the first chamber is

$$A_0=\frac{1}{2}\int_{-\frac{t}{n}}^{t}\left[\mathbf{x}_{h,n+1}(\phi),\mathbf{x}'_{h,n+1}(\phi)\right]\,\mathrm{d}\phi-\frac{1}{2}\int_0^t\left[\mathbf{x}_{h,n}(\phi),\mathbf{x}'_{h,n}(\phi)\right]\,\mathrm{d}\phi$$

$$=\frac{(n-1)t}{n}+\frac{\sin t}{n(n+1)}-\frac{n^2}{n+1}\sin\left(\frac{t}{n}\right),$$

the area of the second chamber is

$$A_1=\frac{1}{2}\int_t^{2\pi+\frac{2\pi-t}{n}}\left[\mathbf{x}_{h,n+1}(\phi),\mathbf{x}'_{h,n+1}(\phi)\right]\,\mathrm{d}\phi-\frac{1}{2}\int_t^{2\pi}\left[\mathbf{x}_{h,n}(\phi),\mathbf{x}'_{h,n}(\phi)\right]\,\mathrm{d}\phi$$

$$=(n-1)\left(\pi-\frac{t}{n}\right)-\frac{\sin t}{n(n+1)}+\frac{1}{2}\sin\left(\frac{t}{n}\right)+\frac{n(n-1)}{2(n+1)}\sin\left(\frac{t-2\pi}{n}\right),$$

and the area $A_2,\ldots,A_n$of the remaining chambers is given by

$$A_k=\frac{1}{2}\int_{2(k-1)\pi+\frac{2(k-1)\pi-t}{n}}^{2k\pi+\frac{2k\pi-t}{n}}\left[\mathbf{x}_{h,n+1}(\phi),\mathbf{x}'_{h,n+1}(\phi)\right]\,\mathrm{d}\phi$$

$$-\frac{1}{2}\int_{2(k-1)\pi}^{2k\pi}\left[\mathbf{x}_{h,n}(\phi),\mathbf{x}'_{h,n}(\phi)\right]\,\mathrm{d}\phi$$

$$=\frac{2(n-1)\pi}{n}-\frac{n^2}{n+1}\left(\sin\left(\frac{2k\pi-t}{n}\right)+\sin\left(\frac{t-2(k-1)\pi}{n}\right)\right).$$

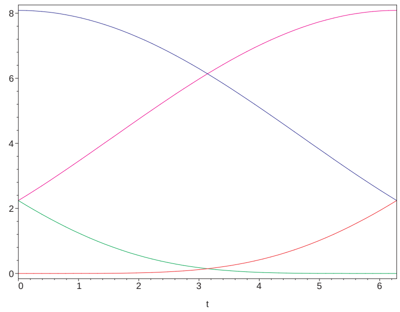


Figure 3: The areas of the chambers in the hypocycloid design in the case $n = 3$, A_0 is red, A_1 is green, A_2 is blue, and A_3 is magenta.

Finally, the area enclosed by a hypocycloid obtained by rolling a circle with radius 1 inside a circle with radius n is

$$\frac{1}{2}\int_0^{2n\pi} \left[\mathbf{x}_{h,n}(\phi), \mathbf{x}'_{h,n}(\phi)\right] \, \mathrm{d}\phi = \frac{1}{2}\frac{(n-1)(n-2)}{n}2n\pi = (n-1)(n-2)\pi. \quad \square$$

We now want to offset these two hypocycloids, but we have a problem at the cusps. If we offset the distance d then we get a singularity (normally a cusp) when the radius of curvature $\rho = 1/\kappa$ equals d. We have

$$\begin{aligned}
\mathbf{x}'_{h,n} &= \frac{n-1}{n}\mathbf{f}\left(\frac{\phi}{n}\right) + \frac{1-n}{n}\mathbf{f}\left(\frac{(1-n)\phi}{n}\right), \\
\left|\mathbf{x}'_{h,n}\right| &= \frac{n-1}{n}\sqrt{2}\sqrt{1-\cos\phi}, \\
\mathbf{x}''_{h,n} &= -\frac{n-1}{n^2}\mathbf{e}\left(\frac{\phi}{n}\right) - \frac{(1-n)^2}{n^2}\mathbf{e}\left(\frac{(1-n)\phi}{n}\right), \\
\left[\mathbf{x}'_{h,n}, \mathbf{x}''_{h,n}\right] &= \frac{(1-n)^2}{n^3}\left(\mathbf{e}\left(\frac{\phi}{n}\right) - \mathbf{e}\left(\frac{(1-n)\phi}{n}\right)\right) \\
&\qquad \cdot \left(\mathbf{e}\left(\frac{\phi}{n}\right) + (n-1)\mathbf{e}\left(\frac{(n-1)\phi}{n}\right)\right) \\
&= \frac{(n-1)^2}{n^3}(1 + (n-1)\cos\phi - \cos\phi - (n-1)) \\
&= \frac{(n-1)^2(n-2)}{n^3}(\cos\phi - 1), \\
\kappa &= \frac{(n-1)^2(n-2)}{n^3}(\cos\phi - 1)\left(\frac{(n-1)}{n}\sqrt{2}\sqrt{1-\cos\phi},\right)^{-3} \\
&= -\frac{(n-2)}{(n-1)2\sqrt{2}\sqrt{1-\cos\phi}},
\end{aligned}$$

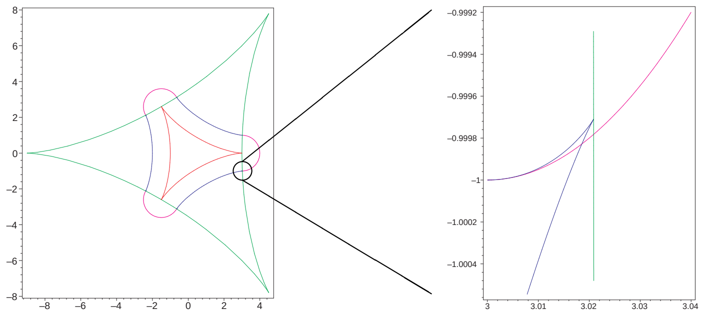


Figure 4: To the left the offset of a hypocycloid with half circles added at the cusps – "the classical design"?. The green curve is the evolute, i.e., the locus of the centre of curvature. To the right we have zoomed in on the transition from the hypocycloid offset to the half circle. Observe the different scalings on the axes.

$$\rho = \frac{1}{\kappa} = -\frac{(n-1)2\sqrt{2}\sqrt{1-\cos\phi}}{n-2}.$$

So $\rho^2 = d^2$ if and only if

$$\cos\phi = 1 - \frac{(n-2)^2 d^2}{8(n-1)^2} = \frac{8(n-1)^2 - (n-2)^2 d^2}{8(n-1)^2} \tag{20}$$

If $d < |\rho|$ then the offset in distance d of a curve $\mathbf{x}(\phi)$ is given by

$$\mathbf{y}_d(\phi) = \mathbf{x}(\phi) \pm d\mathbf{n}(\phi), \tag{21}$$

where $\mathbf{n}$ is the normal. We want to offset on the outside so we consider the offsets

$$\mathbf{y}_{h,n,d}(\phi) = \mathbf{x}_{h,n}(\phi) - d\mathbf{n}(\phi), \qquad d \geq 0. \tag{22}$$

In Figure 4 we have plotted the offset, or rather the curve (21), of a hypocycloid and added a half circle at the cusps. From a first glance everything seems fine but a closer look reveals a cusp where the offset intersects the evolute. It is of course possible to use the true offset, but the true offset of a matching pair of inner and outer curves (pump design) will not match up. The inner curve will be slightly too large and the outer slightly too small near the cusps, and they will consequently no longer touch but intersect during the motion, see Figure 5. This error is in practise compensated for by the rubber. If we try to remedy the situation by removing the overlap then we will produce gaps, i.e., leakage, at other times during the motion.

Even though an offset has cusps it might still be interesting to know, or rather to estimate, the area. We divide the area enclosed by the offset into three areas. The first is the area of region enclosed by the hypocycloid, the second is the area between the hypocycloid and the offset (22) and the third is the area enclosed by

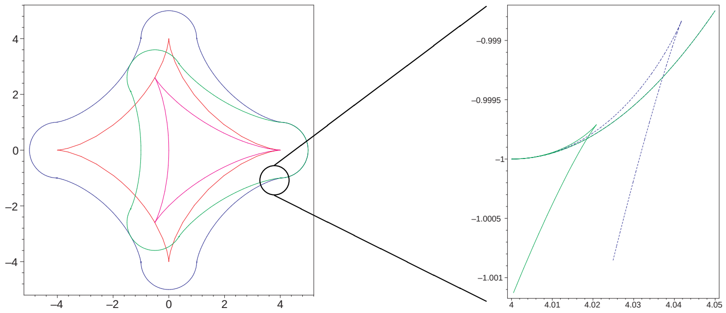


Figure 5: To the left the classical design at a critical time. To the right we have zoomed in on the problematic area. Observe the different scalings on the axes.

the halfcircles added at the cusps. The only unknown area is the second. If we let the offset grows from zero to the final value d, then it sweeps out the second area and it can be found by integrating the length of the offset from zero to d. If not for the cusp this would be true. But if we look at the right hand picture in Figure 4 then we realise that the area above the part of the offset, the blue curve, before the cusp is covered twice and a third time by the half circle. Luckely, as we shall see below it is very easy to let the length of the offset until the cusp to be counted negative, and then if in this way calculate the area with sign then the problematic area is covered only once. There is still a problem with the very small “triangular shaped” area between the offset and the half circle which is covered twice, but we neglect this. Differentiation of (22) yields

$$\mathbf{y}'_{h,n,d}(\phi) = \mathbf{x}'_{h,n}(\phi) - d\mathbf{n}'(\phi) = \mathbf{x}'_{h,n}(\phi) + d\kappa\frac{\mathrm{d}s}{\mathrm{d}\phi}\mathbf{t}(\phi) = (1 + d\kappa)\mathbf{x}'_{h,n}(\phi).$$

Hence

$$\begin{aligned}|\mathbf{y}'_{h,n,d}(\phi)| &= \left(1 - \frac{d(n-2)}{(n-1)2\sqrt{2}\sqrt{1-\cos\phi}}\right)\frac{n-1}{n}\sqrt{2}\sqrt{1-\cos\phi} \\ &= \left|\frac{n-1}{n}\sqrt{2}\sqrt{1-\cos\phi} - \frac{d(n-2)}{2n}\right|.\end{aligned}$$

We clearly see that $\frac{n-1}{n}\sqrt{2}\sqrt{1-\cos\phi} - \frac{d(n-2)}{2n}$ is negative for $\phi \in [0, \phi_c]$ and also for $\phi \in [2\pi - \phi_c, 2\pi]$, where ϕ_c is the parameter value where the offset has the cusp. It is exactly these parts of the curve we want to be counted negative so the “signed” length of one of the n segments of the offset is

$$L(d) = \int_0^{2\pi} \left(\frac{n-1}{n}\sqrt{2}\sqrt{1-\cos\phi} - \frac{d(n-2)}{2n}\right) \mathrm{d}\phi = \frac{8(n-1)}{n} - \frac{(n-2)\pi d}{n}.$$

So the signed area swept out by one segment of the offset is given by

$$\int_0^d L(t)\,\mathrm{d}t = \frac{8(n-1)d}{n} - \frac{(n-2)\pi d^2}{2n}.$$

So the signed area swept ot by the n segments of the offset together with the area of the n half circles and the area enclosed by the hypocycloid gives all in all

$$(n-1)(n-2)\pi+8(n-1)d-\frac{(n-2)\pi d^2}{2}+\frac{n\pi d^2}{2} = (n-1)(n-2)\pi+8(n-1)d+\pi d^2.$$

Thus for the offset design we have the following areas

$$A_{d,\text{outer}} = n(n-1)\pi + 8nd + \pi d^2, \tag{23}$$
$$A_{d,\text{inner}} = (n-1)(n-2)\pi + 8(n-1)d + \pi d^2, \tag{24}$$
$$A_{d,\text{chamber}} = 2(n-1)\pi + 8d. \tag{25}$$

Similarly, for epicycloids we obtain the following

Theorem 3. *The epicycloid obtained by rolling a circle of radius 1 outside a circle of radius n can be parametrised as*

$$\mathbf{x}_{e,n}(\phi) = (n+1)\mathbf{e}\left(\frac{\phi}{n}\right) - \mathbf{e}\left(\frac{(n+1)\phi}{n}\right), \quad t \in [0, 2n\pi], \tag{26}$$

where ϕ is the distance the small circle has rolled, see Figure 6.

If it is moved by the motion obtained by rolling a circle of radius n inside a circle of radius $n+1$ in such a way that the point of contact between the two circles has speed 1, then at time t the point $\mathbf{x}_{e,n}(t)$ touches the epicycloid $\mathbf{x}_{e,n+1}$ in the point $\mathbf{x}_{e,n+1}(t)$. Furthermore, the $n+1$ cusps on the fixed epicycloid $\mathbf{x}_{e,n+1}$ has contact with $\mathbf{x}_{e,n}$ at the points

$$\mathbf{x}_{e,n}\left(\frac{2kn\pi + t}{n+1}\right), \quad k = 1\dots n. \tag{27}$$

The $n+2$ contact points give $n+2$ chambers which for $t \in [0, 2\pi]$ have areas given by

$$A_0 = \frac{(n+2)t}{n+1} + \frac{\sin t}{n(n+1)} - \frac{(n+1)^2}{n}\sin\left(\frac{t}{n+1}\right) \tag{28}$$
$$A_1 = \frac{(n+2)(2\pi-t)}{n+1} - \frac{\sin t}{n(n+1)} - \frac{(n+1)^2}{n}\sin\left(\frac{2\pi-t}{n}\right) \tag{29}$$

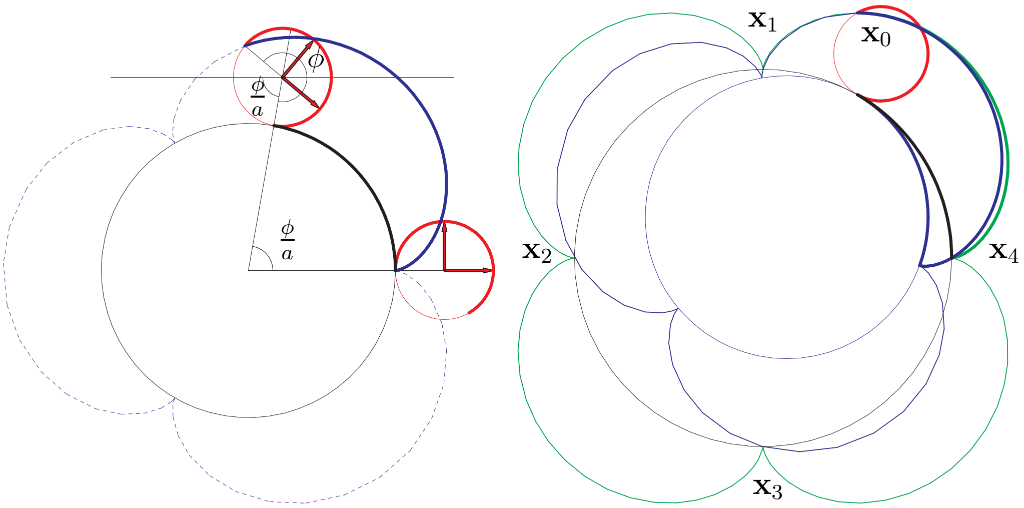


Figure 6: To the left the parametrisation of an epicycloid. To the right the left picture is rolled inside a circle with radius $n+1$ such that the three thick circle segments all have length t.

and for $k = 2, \ldots, n+1$

$$A_k = \frac{2(n+2)\pi}{n+1} + \frac{(n+1)^2}{n} \left(\sin\left(\frac{t-2k\pi}{n+1}\right) + \sin\left(\frac{2(k+1)\pi - t}{n+1}\right)\right) \quad (30)$$

see Figure 7. The situation at another time $t \in \mathbb{R}$ *is given by symmetry. The total area is*

$$A_{total} = \sum_{k=0}^{n+1} A_k = 2(n+2)\pi, \quad (31)$$

and the area of the inner part, of the outer part, and of the circumscribed circle is

$$A_{inner} = (n+1)(n+2)\pi, \quad (32)$$
$$A_{outer} = (n+2)(n+3)\pi, \quad (33)$$
$$A_{circumcircle} = (n+3)^2\pi. \quad (34)$$

Proof. Consider Figure 6. When a circle with radius b has rolled the distance ϕ outside a circle with radius a the centre is at $(a+b)\mathbf{e}(\phi/a)$, and it has rotated through the angle $(a+b)\phi/(ab)$. So if a point has coordinates (x, y) with respect to a coordinate system with origin in the centre of the moving circle, then it is moved to

$$\mathbf{x}(\phi) = (a+b)\mathbf{e}\left(\frac{\phi}{a}\right) + x\mathbf{e}\left(\frac{(a+b)\phi}{ab}\right) + y\mathbf{f}\left(\frac{(a+b)\phi}{ab}\right). \quad (35)$$

Letting $a = n$, $b = 1$, and $(x, y) = (-1, 0)$ shows (26).

Just as before we now subject the epicycloid $\mathbf{x}_{e,n}$ to the motion obtained by letting the circle with radius n roll inside a circle of radius $n + 1$. Clearly, at time t the point $\mathbf{x}_{e,n}(t)$ on the moving epicycloid and the point $\mathbf{x}_0(t) = \mathbf{x}_{e,n+1}(t)$ on the fixed epicycloid coincide, see Figure 6. Just as for hypocycloids the tangents of the two epicycloids coincide too. Thus, the epicycloid $\mathbf{x}_{e,n+1}$ is an envelope.

If we switch the role of the moving and fixed plane then the motion of the point with coordinates (x, y) in the original fixed plane, is given by

$$\mathbf{x}(\phi) = -\mathbf{e}\left(\frac{t}{n}\right) + x\mathbf{e}\left(\frac{t}{n(n+1)}\right) + y\mathbf{f}\left(\frac{t}{n(n+1)}\right).$$

The $n+1$ cusps on the epicycloid $\mathbf{x}_{e,n+1}$ have coordinates $(n+1)\mathbf{e}(2k\pi/(n+1))$, $k = 1 \ldots n + 1$. Inserting this above shows that they traces the curves

$$\begin{aligned}
\mathbf{x}_k(t) &= -\mathbf{e}\left(\frac{t}{n}\right) + (n+1)\cos\left(\frac{2k\pi}{n+1}\right)\mathbf{e}\left(\frac{t}{n(n+1)}\right) \\
&\qquad + (n+1)\sin\left(\frac{2k\pi}{n+1}\right)\mathbf{f}\left(\frac{t}{n(n+1)}\right) \\
&= -\mathbf{e}\left(\frac{t}{n}\right) + (n+1)\mathbf{e}\left(\frac{2k\pi}{n+1} + \frac{t}{n(n+1)}\right) \\
&= -\mathbf{e}\left(\frac{t}{n}\right) + (n+1)\mathbf{e}\left(\frac{2kn\pi + t}{n(n+1)}\right)
\end{aligned}$$

letting $t = (n+1)\phi_k - 2kn\pi$ we get

$$\begin{aligned}
&= (n+1)\mathbf{e}\left(\frac{\phi_k}{n}\right) - \mathbf{e}\left(\frac{(n+1)\phi_k}{n} - 2k\pi\right) \\
&= (n+1)\mathbf{e}\left(\frac{\phi_k}{n}\right) - \mathbf{e}\left(\frac{(n+1)\phi_k}{n}\right) \\
&= \mathbf{x}_{e,n}(\phi_k) = \mathbf{x}_{e,n}\left(\frac{2kn\pi + t}{n+1}\right).
\end{aligned}$$

In a manner similar to case of the hypocycloids it can be shown that we have found all contact points.

Now we only need to find the area of the chambers during the motion. Due to the symmetry it is enough to investigate the design for $t \in [0, 2\pi]$. For such a t the contact points are on the outer part given by the parameter values t and $2k\pi$, where $k = 1, \ldots, n + 1$. The corresponding points on the inner part are given by the parameter values t and $(2kn\pi + t)/(n + 1)$, where $k = 1, \ldots, n + 1$. In both cases $k = 0$ gives the same point as $k = n + 1$. The rest is again a simple integration which we omit. □

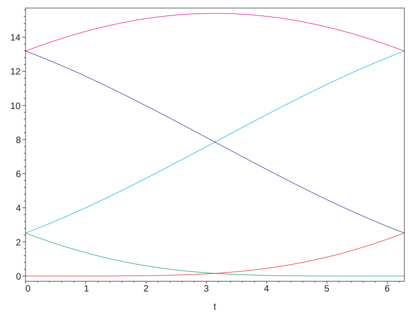


Figure 7: The areas of the chambers in the epicycloid design in the case $n = 3$, A_0 is red, A_1 is green, A_2 is blue, A_3 is magenta, and A_4 is cyan.

Just as in the case of the hypocycloids the cusps of the epicycloids prevent smooth offsets. We have not calculated the areas of the offset construction, but it could be done just as in the case of hypocycloids.

4.2 Composition of alternating hypocycloids and epicycloids

Recall that $\mathbf{x}_{h,n}$ parameterises the hypocycloid obtained by letting a circle with radius 1 roll inside a circle with radius n and that $\mathbf{x}_{e,n}$ parameterises the epicycloid obtained by letting a circle with radius 1 roll outside a circle with radius n. By mimicking the proofs of Theorem 2 and Theorem 3 we can show

Lemma 4. *If the hypocycloid $\mathbf{x}_{h,2n}$ undergoes the motion obtained by letting a circle of radius $2n$ roll inside a circle with radius $2n+2$ in such a way that the point of contact between the two circles has speed 1, then at time t the point $\mathbf{x}_{h,2n}(t)$ touches the hypocycloid $\mathbf{x}_{h,2n+2}$ in the point $\mathbf{x}_{h,2n+2}(t)$.*

Likewise, if the epicycloid $\mathbf{x}_{e,2n}$ undergoes the same motion, then at time t the point $\mathbf{x}_{e,2n}(t)$ touches the epicycloid $\mathbf{x}_{e,2n+2}$ in the point $\mathbf{x}_{e,2n+2}(t)$.

It can also be shown that the two hypocycloids, or epicycloids, only touch in the point given in Lemma 4. Furthermore, the moving cycloids consists of $2n$ arcs and the fixed consists of $2n+2$ arcs and if a point of contact lies on an even arc on one cycloid then so does it on the other. We can now take the odd arcs on the epicycloids and the even arcs on the hypocycloids and obtain the situation in Figure 8. Looking at that figure it seems there are more contact points than the one given in Lemma 4. It is furthermore seen that the extra contact points are on the epicycloid arcs on the inner part and on the hypocycloid arcs on the outer part. At a point of tangential contact the tangents of the two parts are parallel and also parallel to the velocity field at the contact point.

We start by finding the points on the outer hypocycloid and the inner epicycloid where the tangent is parallel to the velocity field or equivalently is orthogonal

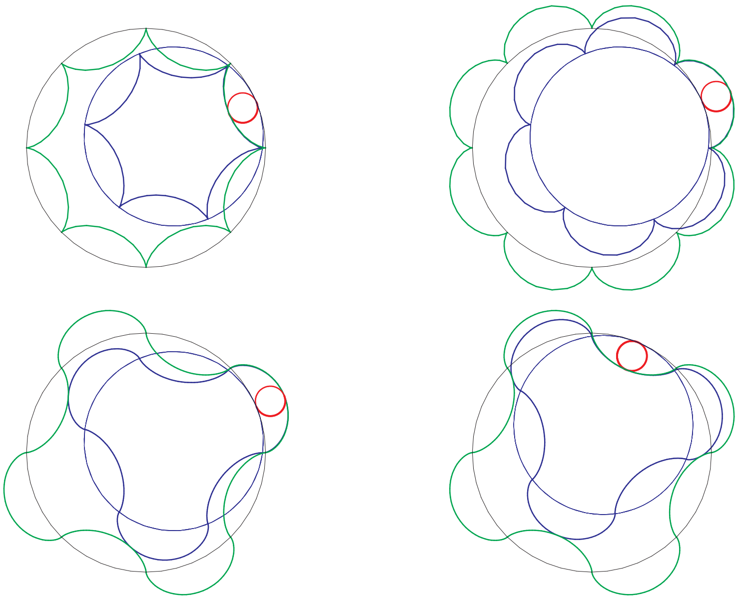


Figure 8: Top left the hypocycloid $\mathbf{x}_{h,2n}$ and top right the epicycloid $\mathbf{x}_{e,2n}$ both following a circle with radius $2n$ rolling inside a circle with radius $2n+2$. In the depicted case $n=3$. Below the composition of the odd arcs from the epicycloids and the even arcs from the hypocycloids.

to the line to the pole. At time t the pole has coordinates $\mathbf{P} = (2n+2)\mathbf{e}\left(\frac{t}{2n+2}\right)$ in the fixed plane. The fixed hypocycloid is given by

$$\mathbf{x}_{h,2n+2}(\phi) = (2n+1)\mathbf{e}\left(\frac{\phi}{2n+2}\right) + \mathbf{e}\left(-\frac{(2n+1)\phi}{2n+2}\right),$$

the derivative is

$$\mathbf{x}'_{h,2n+2}(\phi) = \frac{2n+1}{2n+2}\left(\mathbf{f}\left(\frac{\phi}{2n+2}\right) - \mathbf{f}\left(-\frac{(2n+1)\phi}{2n+2}\right)\right).$$

The line from the pole to a point of contact is orthogonal to the tangent, i.e., we have the equation

$$(\mathbf{x}_{h,2n+2}(\phi) - \mathbf{P}) \cdot \mathbf{x}'_{h,2n+2}(\phi) = 0. \tag{36}$$

Substituting the expressions above we obtain

$$\begin{aligned}
&\frac{2n+2}{2n+1}(\mathbf{x}_{h,2n+2}(\phi) - \mathbf{P}) \cdot \mathbf{x}'_{h,2n+2}(\phi) = \\
&\quad \left((2n+1)\mathbf{e}\left(\frac{\phi}{2n+2}\right) + \mathbf{e}\left(-\frac{(2n+1)\phi}{2n+2}\right) - (2n+2)\mathbf{e}\left(\frac{t}{2n+2}\right)\right) \\
&\qquad \cdot \left(\mathbf{f}\left(\frac{\phi}{2n+2}\right) - \mathbf{f}\left(-\frac{(2n+1)\phi}{2n+2}\right)\right)
\end{aligned}$$

$$= \sin\left(-\frac{(2n+1)\phi}{2n+2} - \frac{\phi}{2n+2}\right) - (2n+2)\sin\left(\frac{t}{2n+2} - \frac{\phi}{2n+2}\right)$$

$$-(2n+1)\sin\left(\frac{\phi}{2n+2} + \frac{(2n+1)\phi}{2n+2}\right) + (2n+2)\sin\left(\frac{t}{2n+2} + \frac{(2n+1)\phi}{2n+2}\right)$$

$$= \sin(-\phi) - (2n+2)\sin\left(\frac{t-\phi}{2n+2}\right) - (2n+1)\sin\phi + (2n+2)\sin\left(\frac{t+(2n+1)\phi}{2n+2}\right)$$

$$= (2n+2)\left(-\sin\phi - \sin\left(\frac{t-\phi}{2n+2}\right) + \sin\left(\frac{t-\phi}{2n+2} + \phi\right)\right)$$

$$= (2n+2)\Bigg(-\sin\phi - \sin\left(\frac{t-\phi}{2n+2}\right)$$

$$+ \cos\left(\frac{t-\phi}{2n+2}\right)\sin\phi + \cos\phi\sin\left(\frac{t-\phi}{2n+2}\right)\Bigg)$$

$$= (2n+2)\left((\cos\phi - 1)\sin\left(\frac{t-\phi}{2n+2}\right) + \left(\cos\left(\frac{t-\phi}{2n+2}\right) - 1\right)\sin\phi\right).$$

Hence (36) is equivalent to

$$(1-\cos\phi)\sin\left(\frac{t-\phi}{2n+2}\right) + \left(1 - \cos\left(\frac{t-\phi}{2n+2}\right)\right)\sin\phi = 0$$

or

$$\frac{\sin\phi}{1-\cos\phi} = \frac{\sin\left(\frac{\phi-t}{2n+2}\right)}{1-\cos\left(\frac{\phi-t}{2n+2}\right)} \tag{37}$$

As the derivative of $\sin\phi/(1-\cos\phi)$ is $-1/(1-\cos\phi)$, which is strictly negative, we have exactly $2n+1$ distinct solutions to (37):

$$\phi_k = \frac{(2n+2)2k\pi - t}{2n+1} = \frac{2k\pi - t}{2n+1} + 2k\pi, \quad k = 0, 1, \ldots, 2n. \tag{38}$$

When $t = 2k\pi$ we see that the k^{th} solution is the cusp $\phi_k = 2k\pi$.

Lemma 5. *If $t \in [2m\pi, 2(m+1)\pi]$, for $m = 0, \ldots, 2n$, then the solutions satisfy*

$$\phi_k \in [2(k-1)\pi, 2k\pi], \quad k = 0, \ldots, m,$$
$$\phi_k \in [2k\pi, 2(k+1)\pi], \quad k = m+1, \ldots, 2n,$$

i.e., the arc given by $\phi \in [2m\pi, 2(m+1)\pi]$ is without contact points.

If $t \in [(4n+2)\pi, (4n+4)\pi]$ then the solutions satisfy

$$\phi_0 \in [-4\pi, -2\pi],$$
$$\phi_k \in [2(k-1)\pi, 2k\pi], \quad k = 1, \ldots, 2n,$$

i.e., the arc given by $\phi \in [-2\pi, 0]$ is without contact points.

Proof. If we let $t = 2(m+\tau)\pi$ where $\tau \in [0,1]$ then

$$\phi_k = \frac{2(k-m-\tau)\pi}{2n+1} + 2k\pi = \left(k + \frac{k-m-\tau}{2n+1}\right) 2\pi,$$

and we have $\phi_k \in [2(k-1)\pi, 2k\pi]$ if and only if

$$-1 \leq \frac{k-m-\tau}{2n+1} \leq 0 \iff k = 0, \ldots, m-1.$$

Likewise, $\phi_k \in [2k\pi, 2(k+1)\pi]$ if and only if

$$0 \leq \frac{k-m-\tau}{2n+1} \leq 1 \iff k = m, \ldots, 2n.$$

Finally, if $t = 2(2n+1+\tau)\pi$ where $\tau \in [0,1]$, then

$$\phi_k = \frac{2k\pi - 2(2n+1+\tau)\pi}{2n+1} + 2k\pi = \left(k - 1 + \frac{k-\tau}{2n+1}\right) 2\pi.$$

We see that $\phi_0 \in [-4\pi, -2\pi]$ and $\phi_k \in [2(k-1)\pi, 2k\pi]$ for $k = 1, \ldots, 2n$. $\square$

In the moving plane the pole has coordinates $\mathbf{Q} = 2n\mathbf{e}\left(\frac{t}{2n}\right)$. The moving epicycloid is given by

$$\mathbf{x}_{e,2n}(\theta) = (2n+1)\mathbf{e}\left(\frac{\theta}{2n}\right) - \mathbf{e}\left(\frac{(2n+1)\theta}{2n}\right),$$

the derivative is

$$\mathbf{x}'_{e,2n}(\theta) = \frac{2n+1}{2n}\left(\mathbf{f}\left(\frac{\theta}{2n}\right) - \mathbf{f}\left(\frac{(2n+1)\theta}{2n}\right)\right).$$

The line from the pole to a point of contact is orthogonal to the tangent, i.e., we have the equation

$$(\mathbf{x}_{e,2n}(\theta) - \mathbf{Q}) \cdot \mathbf{x}'_{e,2n}(\theta) = 0. \tag{39}$$

Substituting the expressions above we obtain

$$\begin{aligned}
&\frac{2n}{2n+1}(\mathbf{x}_{e,2n}(\theta) - \mathbf{Q}) \cdot \mathbf{x}'_{e,2n}(\theta) \\
&= \left((2n+1)\mathbf{e}\left(\frac{\theta}{2n}\right) - \mathbf{e}\left(\frac{(2n+1)\theta}{2n}\right) - 2n\mathbf{e}\left(\frac{t}{2n}\right)\right) \\
&\qquad\qquad \cdot \left(\mathbf{f}\left(\frac{\theta}{2n}\right) - \mathbf{f}\left(\frac{(2n+1)\theta}{2n}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sin\left(\frac{(2n+1)\theta}{2n} - \frac{\theta}{2n}\right) - 2n\sin\left(\frac{t}{2n} - \frac{\theta}{2n}\right) \\
&\quad - (2n+1)\sin\left(\frac{\theta}{2n} - \frac{(2n+1)\theta}{2n}\right) + 2n\sin\left(\frac{t}{2n} - \frac{(2n+1)\theta}{2n}\right) \\
&= -\sin\theta - 2n\sin\left(\frac{t-\theta}{2n}\right) - (2n+1)\sin(-\theta) + 2n\sin\left(\frac{t-\theta}{2n} - \theta\right) \\
&= 2n\left(\sin\theta - \sin\left(\frac{t-\theta}{2n}\right) + \sin\left(\frac{t-\theta}{2n} - \theta\right)\right) \\
&= 2n\left(\sin\theta - \sin\left(\frac{t-\theta}{2n}\right) + \cos\theta\sin\left(\frac{t-\theta}{2n}\right) - \cos\left(\frac{t-\theta}{2n}\right)\sin\theta\right) \\
&= 2n\left(\left(1 - \cos\left(\frac{t-\theta}{2n}\right)\right)\sin\theta - (1-\cos\theta)\sin\left(\frac{t-\theta}{2n}\right)\right).
\end{aligned}
$$

Hence (39) is equivalent to

$$
\left(1 - \cos\left(\frac{t-\theta}{2n}\right)\right)\sin\theta = (1-\cos\theta)\sin\left(\frac{t-\theta}{2n}\right)
$$

or

$$
\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\left(\frac{t-\theta}{2n}\right)}{1-\cos\left(\frac{t-\theta}{2n}\right)} \tag{40}
$$

Just as before we ignore the solutions $\theta = 2k\pi$ that correspond to the cusps and are left with $2n+1$ distinct solutions

$$
\theta_k = \frac{t + 4nk\pi}{2n+1} = \frac{t - 2k\pi}{2n+1} + 2k\pi \quad k = 0, \ldots, 2n. \tag{41}
$$

We have again that when $t = 2k\pi$ then the k^{th} solution is the cusp $\theta_k = 2k\pi$.

Lemma 6. *If $t \in [2m\pi, 2(m+1)\pi]$, for $m = 0, \ldots, 2n$, then the solutions satisfy*

$$
\begin{aligned}
&\theta_k \in [2k\pi, 2(k+1)\pi], \quad k = 0, \ldots, m, \\
&\theta_k \in [2(k-1)\pi, 2k\pi], \quad k = m+1, \ldots, 2n,
\end{aligned}
$$

i.e., on the arc given by $\theta \in [2m\pi, 2(m+1)\pi]$ there are two contact points.

If $t \in [(4n+2)\pi, (4n+4)\pi]$ then the solutions satisfy

$$
\begin{aligned}
&\theta_0 \in [2\pi, 4\pi], \\
&\theta_k \in [2k\pi, 2(k+1)\pi], \quad k = 1, \ldots, 2n,
\end{aligned}
$$

i.e., the on the arc given by $\theta \in [0, 2\pi]$ there are two contact points.

Proof. If we let $t = 2(m+\tau)\pi$ where $\tau \in [0,1]$ then

$$\theta_k = \frac{2(\tau + m - k)\pi}{2n+1} + 2k\pi = \left(k + \frac{m-k+\tau}{2n+1}\right)2\pi,$$

and we have $\theta_k \in [2k\pi, 2(k+1)\pi]$ if and only if

$$0 \leq \frac{m-k+\tau}{2n+1} \leq 1 \iff k = 0, \ldots, m.$$

Likewise $\theta_k \in [2(k-1)\pi, 2k\pi]$ if and only if

$$-1 \leq \frac{m-k+\tau}{2n+1} \leq 0 \iff k = m+1, \ldots, 2n.$$

If we let $t = 2(2n+1+\tau)\pi$ where $\tau \in [0,1]$ then

$$\theta_k = \frac{2(2n+1+\tau)\pi - 2k\pi}{2n+1} + 2k\pi = \left(k + 1 + \frac{\tau - k}{2n+1}\right)2\pi,$$

and we can see that $\theta_0 \in [2\pi, 4\pi]$ and $\theta_k \in [2k\pi, 2(k+1)\pi]$ for $k = 1, \ldots, 2n$. $\square$

What is left to show is that the point $\mathbf{x}_{e,2n}(\theta_k)$ on the inner part at time t coincide with the point $\mathbf{x}_{h,2n+2}(\phi_k)$ on the outer part. The difference between the two points is at time t

$$\begin{aligned}
&2\mathbf{e}\left(\frac{t}{2n+2}\right) + \mathbf{R}\left(\frac{-t}{2n(n+1)}\right)\mathbf{x}_{e,2n}(\theta_k) - \mathbf{x}_{h,2n+2}(\phi_k) \\
&= 2\mathbf{e}\left(\frac{t}{2n+2}\right) + \mathbf{R}\left(\frac{-t}{2n(n+1)}\right)\left((2n+1)\mathbf{e}\left(\frac{\theta_k}{2n}\right) - \mathbf{e}\left(\frac{(2n+1)\theta_k}{2n}\right)\right) \\
&\qquad - (2n+1)\mathbf{e}\left(\frac{\phi_k}{2n+2}\right) - \mathbf{e}\left(-\frac{(2n+1)\phi_k}{2n+2}\right) \\
&= 2\mathbf{e}\left(\frac{t}{2n+2}\right) + (2n+1)\mathbf{e}\left(\frac{t+4nk\pi}{2n(2n+1)} - \frac{t}{2n(n+1)}\right) \\
&\qquad - \mathbf{e}\left(\frac{(2n+1)(t+4nk\pi)}{2n(2n+1)} - \frac{t}{2n(n+1)}\right) \\
&\qquad - (2n+1)\mathbf{e}\left(\frac{-t+(2n+2)2k\pi}{(2n+1)(2n+2)}\right) - \mathbf{e}\left(\frac{t-(2n+2)2k\pi}{2n+2}\right) \\
&= 2\mathbf{e}\left(\frac{t}{2n+2}\right) + (2n+1)\mathbf{e}\left(\frac{-t}{(2n+1)(n+1)} + \frac{2k\pi}{2n+1}\right) \\
&\qquad - \mathbf{e}\left(\frac{t}{2n+2} + 2k\pi\right) \\
&\qquad - (2n+1)\mathbf{e}\left(\frac{-t}{(2n+1)(2n+2)} + \frac{2k\pi}{2n+1}\right) - \mathbf{e}\left(\frac{t}{2n+2} - 2k\pi\right) = \mathbf{0}.
\end{aligned}$$

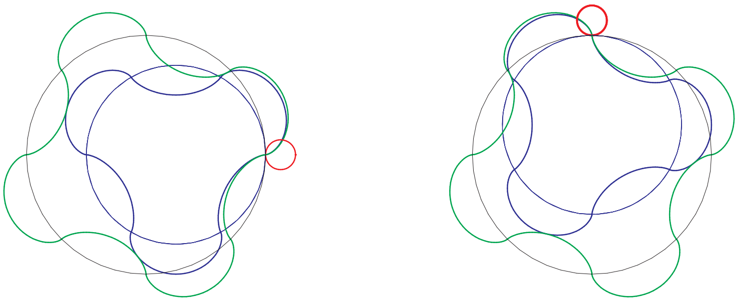


Figure 9: After rolling the distance 4π the configuration is up to a rotation by $\frac{2}{n+1}\pi$ the same as the initial configuration.

We now give a complete description of the design when t varies. we need only consider $t \in [0, 4\pi]$.

Theorem 7. *Consider the design of alternating epi and hypocycloids where the inner part has n teeth and the outer $n+1$ teeth. The inner part is given by*

$$\mathbf{x}(\theta) = \begin{cases} \mathbf{x}_{e,2n}(\theta), & \theta \in [0, 2\pi] \cup [4\pi, 6\pi] \cup \cdots \cup [(4n-4)\pi, (4n-2)\pi], \\ \mathbf{x}_{h,2n}(\theta), & \theta \in [2\pi, 4\pi] \cup [6\pi, 8\pi] \cup \cdots \cup [(4n-2)\pi, 4n\pi], \end{cases}$$

and the outer by

$$\mathbf{y}(\phi) = \begin{cases} \mathbf{x}_{e,2n+2}(\phi), & \phi \in [0, 2\pi] \cup [4\pi, 6\pi] \cup \cdots \cup [4n\pi, (4n+2)\pi], \\ \mathbf{x}_{h,2n+2}(\phi), & \phi \in [2\pi, 4\pi] \cup [6\pi, 8\pi] \cup \cdots \cup [(4n+2)\pi, (4n+4\pi]. \end{cases}$$

The inner part now undergoes the motion generated by letting a circle of radius $2n$ roll inside a circle of radius $2n+2$. Due to the symmetry we only need to investigate the case $t \in [0, 4\pi]$. If we let

$$\theta_k = 2k\pi - \frac{2k\pi - t}{2n+1} \quad \text{and} \quad \phi_k = 2k\pi + \frac{2k\pi - t}{2n+1} \tag{42}$$

and if $t \in [0, 2\pi]$ then we have $n+2$ contact points with parameter values

$$(\theta_0, \phi_0), (t, t), (\theta_1, \phi_1), (\theta_3, \phi_3), \ldots, (\theta_{2n-1}, \phi_{2n-1}), \tag{43}$$

on the inner and outer part respectively and if $t \in [2\pi, 4\pi]$ we have $n+1$ contact points with parameter values

$$(\theta_0, \phi_0), (t, t), (\theta_3, \phi_3), (\theta_5, \phi_5), \ldots, (\theta_{2n-1}, \phi_{2n-1}). \tag{44}$$

The area of the chamber between the contact points (θ_0, ϕ_0) and (t, t) is

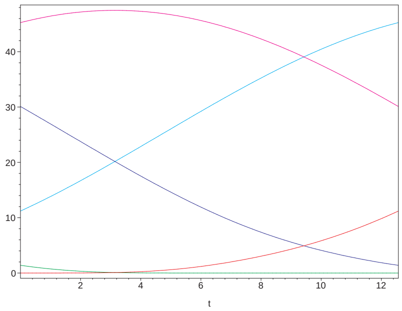


Figure 10: The areas of the chambers in the composed design in the case $n = 3$, A_0 is red, A_1 is green, A_2 is blue, A_3 is magenta, and A_4 is cyan.

$$A_0 = 2t + \frac{\sin t}{2n(n+1)} - \frac{(2n+1)^2}{2n(n+1)} \sin\left(\frac{t}{2n+1}\right), \quad t \in [0, 4\pi], \tag{45}$$

the area of the chamber between the contact points (t, t) *and* (θ_1, ϕ_1) *is*

$$A_1 = \begin{cases} 2(2\pi - t) - \frac{\sin t}{2n(n+1)} - \frac{(2n+1)^2}{2n(n+1)} \sin\left(\frac{2\pi - t}{2n+1}\right), & t \in [0, 2\pi], \\ 0, & t \in [2\pi, 4\pi], \end{cases} \tag{46}$$

the area of the chamber before the contact point (θ_3, ϕ_3) *is*

$$A_2 = \begin{cases} 8\pi + \frac{(2n+1)^3}{2n(n+1)} \left(\sin\left(\frac{2\pi - t}{2n+1}\right) - \sin\left(\frac{6\pi - t}{2n+1}\right)\right), & t \in [0, 2\pi], \\ 12\pi - 2t - \frac{\sin t}{2n(n+1)} - \frac{(2n+1)^3}{2n(n+1)} \sin\left(\frac{6\pi - t}{2n+1}\right), & t \in [2\pi, 4\pi], \end{cases} \tag{47}$$

and the area of the chamber between the contact points $(\theta_{2\ell-3}, \phi_{2\ell-3})$ *and* $(\theta_{2\ell-1}, \phi_{2\ell-1})$ *is for* $\ell = 3, \ldots, n+1$

$$A_\ell = 8\pi + \frac{(2n+1)^3}{2n(n+1)} \left(\sin\left(\frac{2(2\ell-3)\pi - t}{2n+1}\right) - \sin\left(\frac{2(2\ell-1)\pi - t}{2n+1}\right)\right). \tag{48}$$

The total area is

$$A_{total} = \sum_{\ell=0}^{n} A_\ell = 4(2n+1)\pi. \tag{49}$$

Proof. We have already seen most of the theorem. We only need to prove equations (45) – (49). This is a tedious exercise in integration which we omit. The total area is easy to find. We have

$$\mathbf{x}_{e,2n}(\phi) = (2n+1)\mathbf{e}\left(\frac{\phi}{2n}\right) - \mathbf{e}\left(\frac{(2n+1)\phi}{2n}\right),$$

$$
\begin{aligned}
\mathbf{x}'_{e,2n}(\phi) &= \frac{2n+1}{2n}\mathbf{f}\left(\frac{\phi}{2n}\right) - \frac{2n+1}{2n}\mathbf{f}\left(\frac{(2n+1)\phi}{2n}\right), \\
[\mathbf{x}_{e,2n}(\phi), \mathbf{x}_{e,2n}(\phi)] &= \frac{2n+1}{2n}\left((2n+1)\mathbf{f}\left(\frac{\phi}{2n}\right) - \mathbf{f}\left(\frac{(2n+1)\phi}{2n}\right)\right) \\
&\quad \cdot \left(\mathbf{f}\left(\frac{\phi}{2n}\right) - \mathbf{f}\left(\frac{(2n+1)\phi}{2n}\right)\right) \\
&= \frac{2n+1}{2n}\left((2n+1) - (2n+1)\cos\phi - \cos\phi + 1\right) \\
&= \frac{(2n+1)(2n+2)(1-\cos\phi)}{2n},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{x}_{h,2n}(\phi) &= (2n-1)\mathbf{e}\left(\frac{\phi}{2n}\right) + \mathbf{e}\left(\frac{(1-2n)\phi}{2n}\right), \\
\mathbf{x}'_{h,2n}(\phi) &= \frac{2n-1}{2n}\mathbf{f}\left(\frac{\phi}{2n}\right) - \frac{2n-1}{2n}\mathbf{f}\left(\frac{(1-2n)\phi}{2n}\right), \\
[\mathbf{x}_{h,2n}(\phi), \mathbf{x}_{h,2n}(\phi)] &= \frac{2n-1}{2n}\left((2n-1)\mathbf{f}\left(\frac{\phi}{2n}\right) + \mathbf{f}\left(\frac{(1-2n)\phi}{2n}\right)\right) \\
&\quad \cdot \left(\mathbf{f}\left(\frac{\phi}{2n}\right) - \mathbf{f}\left(\frac{(1-2n)\phi}{2n}\right)\right) \\
&= \frac{2n-1}{2n}\left((2n-1) - (2n-1)\cos\phi - \cos\phi - 1\right) \\
&= \frac{(2n-1)(2n-2)(1-\cos\phi)}{2n}.
\end{aligned}
$$

The area of the inner part is

$$
\begin{aligned}
A_{\text{inner}} &= \frac{n}{2}\int_0^{2\pi} [\mathbf{x}_{e,2n}(\phi), \mathbf{x}'_{e,2n}(\phi)]\,\mathrm{d}\phi + \frac{n}{2}\int_0^{2\pi} [\mathbf{x}_{h,2n}(\phi), \mathbf{x}'_{e,2n}(\phi)]\,\mathrm{d}\phi \\
&= \frac{(2n+1)(2n+2)\pi}{2} + \frac{(2n-1)(2n-2)\pi}{2} = (4n^2+2)\pi.
\end{aligned}
$$

The area of the outer part is obviously

$$
A_{\text{outer}} = (4(n+1)^2 + 2)\pi,
$$

and the total area is

$$
A_{total} = A_{\text{outer}} - A_{\text{inner}} = (4(n+1)^2 - 4n^2)\pi = 4(2n+1)\pi. \qquad \square
$$

4.3 Hypotrochoids and epitrochoids

Consider once more the motion generated by a circle of radius b rolling inside a circle of radius a. If we follow the motion of a point not on the perimeter of the rolling circle, then the orbit is a so called hypotrochoid, see Figure 11. If $a = n$

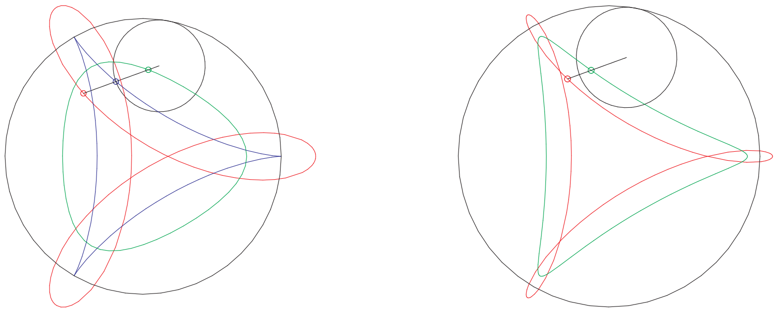


Figure 11: To the left two hypotrochoids ($r = 1/4, 7/4$) and the hypocycloid ($r = 1$). To the right two hypotrochoids ($r = 3/4, 5/4$).

and $b = 1$ and the point we follow has coordinates $(r, 0)$, with $r \le 1$, then the hypotrochoid is parametrised by

$$\mathbf{x}_{h,n,r}(\phi) = (n-1)\mathbf{e}\left(\frac{\phi}{n}\right) + r\mathbf{e}\left(\frac{(1-n)\phi}{n}\right), \tag{50}$$

where ϕ is the distance the small circle has rolled. If it is moved by the motion generated by letting a circle with radius cn roll inside a circle of radius $c(n+1)$ then we obtain the following one-parameter family

$$\begin{aligned}\mathbf{x}(\phi,t) &= c\mathbf{e}\left(\frac{t}{c(n+1)}\right) + \mathbf{R}\left(\frac{-t}{cn(n+1)}\right)\left((n-1)\mathbf{e}\left(\frac{\phi}{n}\right) + r\mathbf{e}\left(\frac{(1-n)\phi}{n}\right)\right) \\ &= c\mathbf{e}\left(\frac{t/c}{n+1}\right) + (n-1)\mathbf{e}\left(\frac{\phi}{n} - \frac{t/c}{n(n+1)}\right) + r\mathbf{e}\left(\frac{(1-n)\phi}{n} - \frac{t/c}{n(n+1)}\right),\end{aligned}$$

or by changing the speed,

$$\mathbf{x}(\phi,t) = c\mathbf{e}\left(\frac{t}{n+1}\right) + (n-1)\mathbf{e}\left(\frac{\phi}{n} - \frac{t}{n(n+1)}\right) + r\mathbf{e}\left(\frac{(1-n)\phi}{n} - \frac{t}{n(n+1)}\right). \tag{51}$$

Now

$$\frac{\partial \mathbf{x}}{\partial \phi} = \frac{n-1}{n}\left(\mathbf{f}\left(\frac{\phi}{n} - \frac{t}{n(n+1)}\right) - r\mathbf{f}\left(\frac{(1-n)\phi}{n} - \frac{t}{n(n+1)}\right)\right)$$

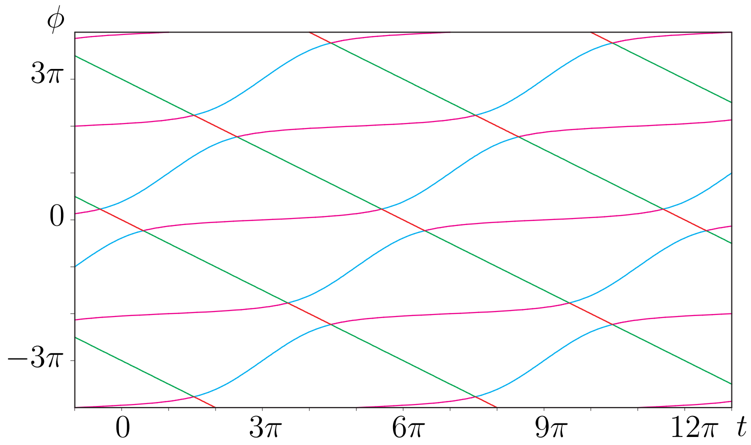


Figure 12: The solution to (56) in the case $c = r = 3/4$, $n = 3$.

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{c}{n+1}\mathbf{f}\left(\frac{t}{n+1}\right) - \frac{n-1}{n(n+1)}\mathbf{f}\left(\frac{\phi}{n} - \frac{t}{n(n+1)}\right) - \frac{r}{n(n+1)}\mathbf{f}\left(\frac{(1-n)\phi}{n} - \frac{t}{n(n+1)}\right).$$

The point where $\frac{\partial \mathbf{x}}{\partial \phi}$ and $\frac{\partial \mathbf{x}}{\partial t}$ are parallel are determined by the equation

$$0 = \left(\mathbf{e}\left(\frac{\phi}{n} - \frac{t}{n(n+1)}\right) - r\mathbf{e}\left(\frac{(1-n)\phi}{n} - \frac{t}{n(n+1)}\right)\right) \cdot \left(c\mathbf{f}\left(\frac{t}{n+1}\right) - \frac{n-1}{n}\mathbf{f}\left(\frac{\phi}{n} - \frac{t}{n(n+1)}\right) - \frac{r}{n}\mathbf{f}\left(\frac{(1-n)\phi}{n} - \frac{t}{n(n+1)}\right)\right)$$
$$= c\sin\left(\frac{\phi - t}{n}\right) + cr\sin\left(\frac{(n-1)\phi + t}{n}\right) - r\sin\phi. \quad (52)$$

If $c = r$ then it is easily seen that $\phi = 2nk\pi - t/(n-1)$ is a solution for each $k \in \mathbb{Z}$ and that $\phi = nk\pi - t/(n-1)$ is a solution for n odd and $k \in \mathbb{Z}$, see Figure 12. But there are other solutions. We define ψ by

$$\cos\psi = \frac{1}{\sqrt{1 - 2r\cos\phi + r^2}}\left(\sin\left(\frac{\phi}{n}\right) + r\sin\left(\frac{(n-1)\phi}{n}\right)\right), \quad (53)$$
$$\sin\psi = \frac{1}{\sqrt{1 - 2r\cos\phi + r^2}}\left(\cos\left(\frac{\phi}{n}\right) - r\cos\left(\frac{(n-1)\phi}{n}\right)\right), \quad (54)$$

and then we can rewrite (52) as

$$c\sqrt{1 - 2r\cos\phi + r^2}\cos\left(\psi + \frac{t}{n}\right) - r\sin\phi = 0, \quad (55)$$

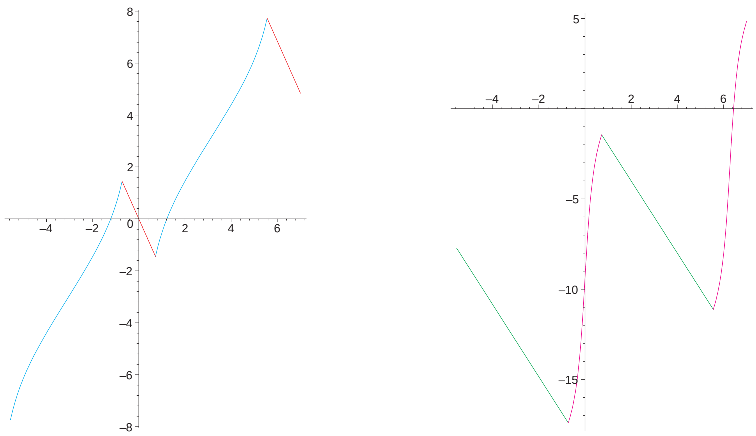


Figure 13: The solutions (57). To the left $t^+(\phi)$ and to the right $t^-(\phi)$. The colours corresponds to the colours in Figure 12.

or

$$\cos\left(\psi + \frac{t}{n}\right) = \frac{r \sin\phi}{c\sqrt{1 - 2r\cos\phi + r^2}}. \tag{56}$$

The derivative of the right hand side vanishes if and only if $\cos\phi = r$ in which case the value is $\pm r/c$. That might indicate that it is a good idea to put $c = r$. Doing this we have the solutions

$$t^\pm = -n\psi \pm n \arccos\left(\frac{\sin\phi}{\sqrt{1 - 2r\cos\phi + r^2}}\right), \tag{57}$$

see Figure 13. The solutions $t = 2n(n-1)k\pi - (n-1)\phi$ are found to be

$$t^+(\phi) = \begin{cases} -(n-1)\phi, & \phi \in [-\phi_0, \phi_0], \\ 2n(n-1)\pi - (n-1)\phi, & \phi \in [2\pi - \phi_0, 2\pi + \phi_0], \end{cases} \tag{58}$$

$$t^-(\phi) = \begin{cases} -2n(n-1)\pi - (n-1)\phi, & \phi \in [-2\pi + \phi_0, -\phi_0], \\ -(n-1)\phi, & \phi \in [\phi_0, 2\pi - \phi_0], \end{cases} \tag{59}$$

The other parts of the solution can now be found by symmetry and the result can be seen in Figure 12. By substituting $\phi = \phi(t)$ or in this case $t = t(\phi)$ in (51) we can find the envelope. The different parts of the curves in Figure 12 gives different parts of the envelope, see Figure 14. In order to prove that the construction forms a number of chambers we need to prove that the magenta and cyan curves in Figure 14 are the same, but this has not been done. We should also check that the construction is without singularities.

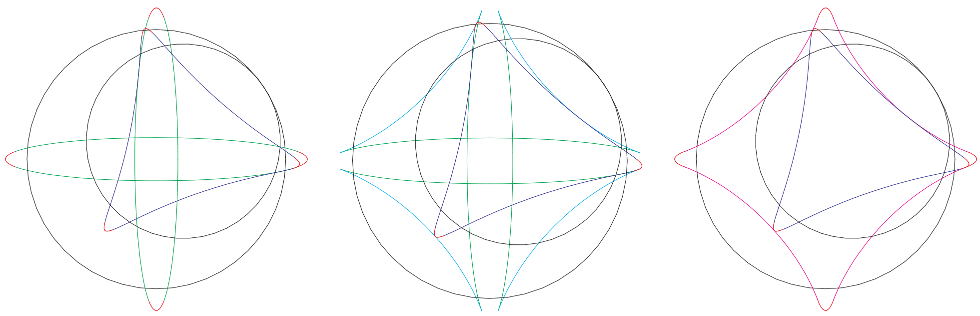


Figure 14: To the left the two inner envelopes obtained by letting $\phi = -t/2$ and $\phi = 3\pi - t/2$ respectively, i.e. by following two neighbouring red and green lines in Figure 12. The red and blue part of the hypotrochoid generates the red and green parts of the envelope, respectively. In the middle the envelope obtained by following a horizontal sequence of cyan and green curves in Figure 12. The envelope is generated by the blue part of the hypotrochoid. To the right the outer envelope obtained by following a sequence of red and magenta curves in Figure 12. The envelope is generated by the red part of the envelope.

From (58) and (59) we find that

$$\arccos\left(\frac{\sin\phi}{\sqrt{1-2r\cos\phi+r^2}}\right) = \begin{cases} -\frac{n-1}{n}\phi+\psi, & \phi\in[-\phi_0,\phi_0], \\ \frac{n-1}{n}\phi-\psi, & \phi\in[\phi_0, 2\pi-\phi_0], \end{cases}$$

and hence

$$t^+ = \begin{cases} -(n-1)\phi, & \phi\in[-\phi_0,\phi_0], \\ (n-1)\phi-2n\psi, & \phi\in[\phi_0,2\pi-\phi_0], \end{cases}$$
$$t^- = \begin{cases} (n-1)\phi-2n\psi, & \phi\in[-\phi_0,\phi_0], \\ -(n-1)\phi, & \phi\in[\phi_0,2\pi-\phi_0]. \end{cases}$$

That is, we have the two solutions $-(n-1)\phi$ and $(n-1)\phi-2n\psi$. Substituting $t=-(n-1)\phi$ in (51) gives the following expression for one of the red parts of the envelope, see Figure 14,

$$\mathbf{y}_1(\phi) = 2r\mathbf{e}\left(\frac{(1-n)\phi}{n+1}\right) + (n-1)\mathbf{e}\left(\frac{2\phi}{n+1}\right), \quad \phi\in[-\phi_0,\phi_0]. \tag{60}$$

Substituting $t=(n-1)\phi-2n\psi$ in (51) yields

$$\mathbf{y}_2(\phi) = r\mathbf{e}\left(-\frac{(n^2+n-2)\phi-2n\psi}{n(n+1)}\right) + r\mathbf{e}\left(\frac{(n-1)\phi-2\psi}{n+1}\right) + (n-1)\mathbf{e}\left(\frac{2\phi+2n\psi}{n(n+1)}\right), \quad \phi\in[\phi_0,2\pi-\phi_0]. \tag{61}$$

The area from the part of the envelope given by $\mathbf{y}_1$ is

$$A_1 = \frac{n+1}{2}\int_{-\phi_0}^{\phi_0} [\mathbf{y}_1(\phi), \mathbf{y}_1'(\phi)]\, \mathrm{d}\phi \\ = 2(n-1)\left((n-1-2r^2)\phi_0 - (n-3)r\sqrt{1-r^2}\right)$$

The area from the part of the envelope given by $\mathbf{y}_2$ is

$$A_2 = \frac{n+1}{2}\int_{\phi_0}^{2\pi-\phi_0} [\mathbf{y}_2(\phi), \mathbf{y}_2'(\phi)]\, \mathrm{d}\phi,$$

but we could not find an analytical expression for this integral. The area of the outer part is $A_{\text{outer}} = A_1 + A_2$ and the area of the inner part is

$$A_{\text{inner}} = \frac{n}{2}\int_0^{2\pi} [\mathbf{x}_{h,n,r}(\phi), \mathbf{x}_{h,n,r}'(\phi)]\, \mathrm{d}\phi = (n-1)(n-1-r^2)\pi.$$

The total area of the chambers is $A_{\text{total}} = A_{\text{outer}} - A_{\text{inner}}$. In the case of $n = 3$ and $r = 3/4$ we find $A_1 = 7\arccos(3/4)$, $A_2 = 32.26$, $A_{\text{outer}} = 37.32$, $A_{\text{inner}} = 23\pi/4$, and $A_{\text{total}} = 14.20$.

It might be of interest to know the curvature of the hypotrochoid. This is a straight forward calculation:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}_{h,n,r}}{\mathrm{d}\phi} &= \frac{n-1}{n}\mathbf{f}\left(\frac{\phi}{n}\right) - r\frac{n-1}{n}\mathbf{f}\left(\frac{(1-n)\phi}{n}\right), \\
\left|\frac{\mathrm{d}\mathbf{x}_{h,n,r}}{\mathrm{d}\phi}\right| &= \frac{n-1}{n}\sqrt{1+r^2-2r\cos\phi}, \\
\frac{\mathrm{d}^2\mathbf{x}_{h,n,r}}{\mathrm{d}\phi^2} &= -\frac{n-1}{n^2}\mathbf{e}\left(\frac{\phi}{n}\right) - r\frac{(n-1)^2}{n^2}\mathbf{e}\left(\frac{(1-n)\phi}{n}\right), \\
\left[\frac{\mathrm{d}\mathbf{x}_{h,n}}{\mathrm{d}\phi}, \frac{\mathrm{d}^2\mathbf{x}_{h,n}}{\mathrm{d}\phi^2}\right] &= \frac{(n-1)^2}{n^3}(1-(n-1)r^2+(n-2)r\cos\phi), \\
\kappa &= \frac{1-(n-1)r^2+(n-2)r\cos\phi}{(n-1)(1+r^2-2r\cos\phi)^{3/2}}, \\
\rho &= \frac{1}{\kappa} = \frac{(n-1)(1+r^2-2r\cos\phi)^{3/2}}{1-(n-1)r^2+(n-2)r\cos\phi}.
\end{aligned}$$

5 The support function

Let h be a real function, and define the vector-valued function

$$\mathbf{x}(\phi) = h(\phi)\mathbf{e}(\phi) + h'(\phi)\mathbf{f}(\phi), \tag{62}$$

$$\begin{aligned}\mathbf{x}'(\phi) &= h'(\phi)\mathbf{e}(\phi) + h(\phi)\mathbf{f}(\phi) + h''(\phi)\mathbf{f}(\phi) - h'(\phi)\mathbf{e}(\phi) && (63)\\ &= (h(\phi) + h''(\phi))\mathbf{f}(\phi), && (64)\end{aligned}$$

we can see that $\mathbf{f}(\phi)$ is the tangent and that $\mathbf{e}(\phi)$ is the normal. We call ϕ for the *normal direction*. The speed, curvature, radius of curvature, and curvature variation is

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\phi} &= h(\phi) + h''(\phi), && (65)\\ \kappa = \frac{\mathrm{d}\phi}{\mathrm{d}s} &= \frac{1}{h(\phi) + h''(\phi)}, && (66)\\ \rho = \frac{1}{\kappa} &= h(\phi) + h''(\phi), && (67)\\ \frac{\mathrm{d}\kappa}{\mathrm{d}s} = \frac{\mathrm{d}\kappa}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s} &= \frac{h'(\phi) + h'''(\phi)}{(h(\phi) + h''(\phi))^3} && (68)\end{aligned}$$

Rotate and translate the curve $\mathbf{x}$:

$$\begin{aligned}\mathbf{X}(\phi, t) &= \mathbf{R}(t)\mathbf{x}(\phi) + c\,\mathbf{e}(\alpha t) && (69)\\ &= h(\phi)\mathbf{e}(\phi + t) + h'(\phi)\mathbf{f}(\phi + t) + c\,\mathbf{e}(\alpha t), && (70)\\ \frac{\partial \mathbf{X}}{\partial \phi} &= (h(\phi) + h''(\phi))\mathbf{f}(\phi + t), && (71)\\ \frac{\partial \mathbf{X}}{\partial t} &= h(\phi)\mathbf{f}(\phi + t) - h'(\phi)\mathbf{e}(\phi + t) + c\alpha\,\mathbf{f}(\alpha t), && (72)\end{aligned}$$

The partial derivatives $\partial\mathbf{X}/\partial\phi$ and $\partial\mathbf{X}/\partial t$ are parallel if and only if

$$\frac{\partial \mathbf{X}}{\partial t} \cdot \mathbf{e}(\phi + t) = 0, \qquad (73)$$

i.e., if and only if

$$-h'(\phi) + c\alpha \sin(\phi + (1 - \alpha)t) = 0,$$

or

$$\sin(\phi + (1 - \alpha)t) = \frac{h'(\phi)}{c\alpha}. \qquad (74)$$

This happens if and only if

$$\phi + (1 - \alpha)t = 2k\pi + \arcsin\left(\frac{h'(\phi)}{c\alpha}\right), \qquad (75)$$

or

$$\phi + (1 - \alpha)t = (2k + 1)\pi - \arcsin\left(\frac{h'(\phi)}{c\alpha}\right), \qquad (76)$$

i.e., we can solve with respect to t and obtain two sets of solutions

$$t^+ = \frac{2k\pi}{1-\alpha} + \frac{1}{1-\alpha}\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) - \phi\right), \tag{77}$$

$$t^- = \frac{(2k+1)\pi}{1-\alpha} - \frac{1}{1-\alpha}\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \phi\right). \tag{78}$$

Observe that

$$\cos(\phi + (1-\alpha)t^+) = \cos\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right)\right) = \sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2} \tag{79}$$

$$\cos(\phi + (1-\alpha)t^-) = \cos\left(\pi - \arcsin\left(\frac{h'(\phi)}{c\alpha}\right)\right) = -\sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2} \tag{80}$$

By differentiation of (74) we find that the derivative of $t^\pm(\phi)$ is

$$\frac{\mathrm{d}t^\pm}{\mathrm{d}\phi} = \frac{1}{1-\alpha}\left(\frac{\pm h''(\phi)}{c\alpha\sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2}} - 1\right). \tag{81}$$

An envelope can be written as $\mathbf{y} = \mathbf{X}(\phi, t)$ where t is given by (77) or (78). The normal of $\mathbf{y}$ is the normal of $\mathbf{x}(\phi)$ rotated through the angle t, i.e., it is $\mathbf{e}(\phi+t)$. So the normal direction of $\mathbf{y}^\pm$ is

$$\phi^\pm = \phi + t^\pm \tag{82}$$

and the support function $h^\pm$ of the envelope $\mathbf{y}^\pm$ is given by

$$\begin{aligned} h^\pm(\phi^\pm) &= \mathbf{y}^\pm(\phi^\pm)\cdot\mathbf{e}(\phi^\pm) = \mathbf{X}(\phi, t^\pm)\cdot\mathbf{e}(\phi + t^\pm) \\ &= h(\phi) + c\cos(\phi + (1-\alpha)t^\pm) = h(\phi) \pm c\sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2}. \end{aligned} \tag{83}$$

The first derivative is

$$\begin{aligned} \frac{\mathrm{d}h^\pm}{\mathrm{d}\phi^\pm} &= h'(\phi^\pm - t^\pm)\left(1 - \frac{\mathrm{d}t^\pm}{\mathrm{d}\phi^\pm}\right) - c\sin(\phi^\pm - \alpha t^\pm)\left(1 - \alpha\frac{\mathrm{d}t^\pm}{\mathrm{d}\phi^\pm}\right) \\ &= h'(\phi)\left(1 - \frac{\mathrm{d}t^\pm}{\mathrm{d}\phi^\pm}\right) - c\frac{h'(\phi)}{c\alpha}\left(1 - \alpha\frac{\mathrm{d}t^\pm}{\mathrm{d}\phi^\pm}\right) \\ &= h'(\phi)\left(1 - \frac{\mathrm{d}t^\pm}{\mathrm{d}\phi^\pm}\right) - h'(\phi)\left(\frac{1}{\alpha} - \frac{\mathrm{d}t^\pm}{\mathrm{d}\phi^\pm}\right) \\ &= \frac{\alpha - 1}{\alpha}h'(\phi). \end{aligned} \tag{84}$$

We can now write the envelope as

$$\mathbf{y}^{\pm} = \left(h(\phi) \pm c\sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2} \right) \mathbf{e}(\phi + t^{\pm}) + \frac{\alpha - 1}{\alpha} h'(\phi) \mathbf{f}(\phi + t^{\pm}), \quad (85)$$

where $t^{\pm}$ is given by (77) or (78). We have

$$\frac{\mathrm{d}t^{\pm}}{\mathrm{d}\phi^{\pm}} = \left(\frac{\mathrm{d}\phi^{\pm}}{\mathrm{d}\phi}\right)^{-1} \frac{\mathrm{d}t^{\pm}}{\mathrm{d}\phi} = \left(1 + \frac{\mathrm{d}t^{\pm}}{\mathrm{d}\phi}\right)^{-1} \frac{\mathrm{d}t^{\pm}}{\mathrm{d}\phi}.$$

So the second derivative of $h^{\pm}$ is

$$\begin{aligned}
\frac{\mathrm{d}^2 h^{\pm}}{\mathrm{d}\phi^{\pm 2}} &= \frac{\alpha - 1}{\alpha} h''(\phi) \left(1 - \frac{\mathrm{d}t^{\pm}}{\mathrm{d}\phi^{\pm}}\right) \\
&= \frac{\alpha - 1}{\alpha} h''(\phi) \left(1 + \frac{\mathrm{d}t^{\pm}}{\mathrm{d}\phi}\right)^{-1} \\
&= \left(\frac{\alpha - 1}{\alpha}\right)^2 \left(\frac{1}{h''(\phi)} \mp \frac{1}{c\alpha^2 \sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2}} \right)^{-1} \qquad (86)
\end{aligned}$$

and the radius of curvature is

$$\rho^{\pm} = h^{\pm} + \frac{\mathrm{d}^2 h^{\pm}}{\mathrm{d}\phi^{\pm 2}}. \quad (87)$$

If we interchange the moving and fixed part of the design and reverse the time then the outer part $\mathbf{y}^{\pm}$ moves like

$$\mathbf{Y}^{\pm}(\phi^{\pm}, t) = -c\mathbf{e}((1 - \alpha)t) + \mathbf{R}(t)\mathbf{y}^{\pm}(\phi^{\pm}), \quad (88)$$

i.e., we have $(c, \alpha) \mapsto (-c, 1 - \alpha)$. The contact points are given by the equation

$$\sin(\phi^{\pm} + \alpha t) = \frac{h^{\pm\prime}(\phi^{\pm})}{-c(1 - \alpha)} = \frac{h'(\phi)}{c\alpha}. \quad (89)$$

and we obtain four sets of solutions

$$\begin{aligned}
t^{\pm +} &= \frac{2\ell\pi}{\alpha} + \frac{1}{\alpha}\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) - \phi^{\pm}\right) \qquad (90) \\
&= \frac{2\ell\pi}{\alpha} + \frac{1}{\alpha}\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) - \phi - t^{\pm}\right), \qquad (91) \\
t^{\pm -} &= \frac{(2\ell + 1)\pi}{\alpha} - \frac{1}{\alpha}\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \phi^{\pm}\right) \qquad (92) \\
&= \frac{(2\ell + 1)\pi}{\alpha} - \frac{1}{\alpha}\left(\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \phi + t^{\pm}\right). \qquad (93)
\end{aligned}$$

Using (77) and (78) we find

$$t^{++} = \frac{2(\ell(1-\alpha)-k)\pi}{\alpha(1-\alpha)} - \frac{1}{1-\alpha}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{\phi}{1-\alpha}, \tag{94}$$

$$t^{-+} = \frac{(2\ell(1-\alpha)-(2k+1))\pi}{\alpha(1-\alpha)} + \frac{2-\alpha}{\alpha(1-\alpha)}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{\phi}{1-\alpha}, \tag{95}$$

$$t^{+-} = \frac{((2\ell+1)(1-\alpha)-2k)\pi}{\alpha(1-\alpha)} - \frac{2-\alpha}{\alpha(1-\alpha)}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{\phi}{1-\alpha}, \tag{96}$$

$$t^{--} = \frac{((2\ell+1)(1-\alpha)-(2k+1))\pi}{\alpha(1-\alpha)} + \frac{1}{1-\alpha}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{\phi}{1-\alpha}, \tag{97}$$

The normal directions are

$$\phi^{++} = \phi + t^{+} + t^{++} = \phi + \frac{2m\pi}{\alpha}, \tag{98}$$

$$\phi^{-+} = \phi + t^{-} + t^{-+} = \phi + \frac{2\arcsin\left(\frac{h'(\phi)}{c\alpha}\right)}{\alpha} + \frac{(2m-1)\pi}{\alpha}, \tag{99}$$

$$\phi^{+-} = \phi + t^{+} + t^{+-} = \phi - \frac{2\arcsin\left(\frac{h'(\phi)}{c\alpha}\right)}{\alpha} + \frac{(2m+1)\pi}{\alpha}, \tag{100}$$

$$\phi^{--} = \phi + t^{-} + t^{--} = \phi + \frac{2m\pi}{\alpha}, \tag{101}$$

where $m = \ell - k$. The support functions are

$$\begin{aligned} h^{\pm\pm} &= h^{\pm}(\phi^{\pm}) \mp c\sqrt{1-\left(\frac{h'(\phi^{\pm})}{-c(1-\alpha)}\right)^2} \\ &= h(\phi) \pm c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2} \mp c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}, \end{aligned} \tag{102}$$

i.e.,

$$h^{++} = h^{--} = h(\phi), \tag{103}$$

$$h^{+-} = h(\phi) + 2c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}, \tag{104}$$

$$h^{-+} = h(\phi) - 2c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}. \tag{105}$$

If $h + h'' > 0$ so h gives the positively curved segment of the inner part and $c > 0$ then it is h^{-+} that gives the missing negatively curved segment of the inner part. The first derivative is

$$\frac{\mathrm{d}h^{\pm\pm}}{\mathrm{d}\phi^{\pm\pm}} = \frac{(1-\alpha)-1}{1-\alpha}\frac{\mathrm{d}h^{\pm}}{\mathrm{d}\theta^{\pm}} = \frac{-\alpha}{1-\alpha}\frac{\alpha-1}{\alpha}h'(\phi) = h'(\phi). \tag{106}$$

The second derivative is

$$\begin{aligned}\frac{\mathrm{d}^2h^{\pm\pm}}{\mathrm{d}\phi^{\pm\pm 2}} &= \left(\frac{(1-\alpha)-1}{1-\alpha}\right)^2\left(\frac{1}{h^{\pm\prime\prime}(\phi^{\pm})} \pm \frac{1}{c(1-\alpha)^2\sqrt{1-\left(\frac{h^{\pm\prime}(\phi^{\pm})}{c(1-\alpha)}\right)^2}}\right)^{-1}\\ &= \left(\frac{\alpha}{1-\alpha}\right)^2\left(\left(\frac{\alpha}{\alpha-1}\right)^2\left(\frac{1}{h''(\phi)} \mp \frac{1}{c\alpha^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}\right)\right.\\ &\qquad\left.\pm\frac{1}{c(1-\alpha)^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}\right)^{-1}\\ &= \left(\frac{1}{h''(\phi)} + \frac{\mp 1 \pm 1}{c\alpha^2\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}}\right)^{-1}.\end{aligned} \tag{107}$$

When the outer part is moving the envelope is parametrised as

$$\mathbf{x}^{++} = \mathbf{x}^{--} = h(\phi)\mathbf{e}\left(\phi + \frac{2m\pi}{\alpha}\right) + h'(\phi)\mathbf{f}\left(\phi + \frac{2m\pi}{\alpha}\right), \tag{108}$$

$$\begin{aligned}\mathbf{x}^{-+} = &\left(h(\phi) - 2c\sqrt{1-\left(\frac{h'(\phi)}{c\alpha}\right)^2}\right)\mathbf{e}\left(\phi + \frac{2}{\alpha}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{(2m-1)\pi}{\alpha}\right)\\ &+ h'(\phi)\mathbf{f}\left(\phi + \frac{2}{\alpha}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{(2m-1)\pi}{\alpha}\right),\end{aligned} \tag{109}$$

$$\mathbf{x}^{+-} = \left(h(\phi) + 2c\sqrt{1 - \left(\frac{h'(\phi)}{c\alpha}\right)^2} \right) \mathbf{e}\left(\phi - \frac{2}{\alpha}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{(2m+1)\pi}{\alpha}\right)$$
$$+ h'(\phi)\mathbf{f}\left(\phi - \frac{2}{\alpha}\arcsin\left(\frac{h'(\phi)}{c\alpha}\right) + \frac{(2m+1)\pi}{\alpha}\right). \tag{110}$$

If a curve $\mathbf{x}(\phi)$ is parameterised by normal direction then

$$[\mathbf{x}, \mathbf{x}'] = [h\mathbf{e} + h'\mathbf{f}, (h + h'')\mathbf{f}] = h(h + h''). \tag{111}$$

So if the inner part has n positively curved segments given by a support function $h : [-\phi_0, \phi_0] \to \mathbb{R}$ then the area is

$$A_{\text{inner}} = \frac{n}{2}\int_{-\phi_0}^{\phi_0} h\left(h + h''\right) \mathrm{d}\phi + \frac{n}{2}\int_{-\phi_0}^{\phi_0} h^{-+}\left(h^{-+} + h^{-+\prime\prime}\right)\frac{\mathrm{d}\phi^{-+}}{\mathrm{d}\phi}\,\mathrm{d}\phi. \tag{112}$$

The outer part then has $n+1$ pair of segments and the area is

$$A_{\text{outer}} = \frac{n+1}{2}\int_{-\phi_0}^{\phi_0} h\left(h^{+} + h^{+\prime\prime}\right)\frac{\mathrm{d}\phi^{+}}{\mathrm{d}\phi}\,\mathrm{d}\phi$$
$$+ \frac{n+1}{2}\int_{-\phi_0}^{\phi_0} h^{-}\left(h^{-} + h^{-\prime\prime}\right)\frac{\mathrm{d}\phi^{-}}{\mathrm{d}\phi}\,\mathrm{d}\phi. \tag{113}$$

5.1 A simple example

Now we consider a segment with endpoints $\mathbf{e}\left(\frac{\pm\pi}{2n}\right)$ where the normals are assumed to be $\pm\phi_0$, where $\phi_0 = \left(\frac{\pi}{2n} + \delta\right)$. The support function is then a function $h : [-\phi_0, \phi_0] \to \mathbb{R}$ and as

$$\mathbf{e}\left(\pm\frac{\pi}{2n}\right) = \cos\delta\,\mathbf{e}\left(\pm\phi_0\right) \mp \sin\delta\,\mathbf{f}\left(\pm\phi_0\right),$$

we have $h\left(\pm\phi_0\right) = \cos\delta$ and $h'\left(\pm\phi_0\right) = \mp\sin\delta$. One simple function that satisfies these equations is

$$h(\phi) = \cos\delta + \frac{2\phi_0}{\pi}\cos\left(\frac{\pi\phi}{2\phi_0}\right)\sin\delta, \tag{114}$$

that has the derivative

$$h'(\phi) = -\sin\left(\frac{\pi\phi}{2\phi_0}\right)\sin\delta. \tag{115}$$

Using the parametrisation (62) we can easily plot the segment, see Figure 16 where $n = 3$ and $\delta = \pi/4$. If $\alpha = -n$ then we have the motion generated by a circle of radius cn rolling inside a circle of radius $c(n+1)$ and if $c = \frac{\sin\delta}{n}$ then (77) and (78) reads

$$t^+ = \frac{2k\pi}{n+1} - \frac{(2\phi_0 - \pi)\phi}{2(n+1)\phi_0}, \qquad t^- = \frac{(2k+1)\pi}{n+1} - \frac{(2\phi_0 + \pi)\phi}{2(n+1)\phi_0}.$$

We have plotted the solutions in Figure 15. The envelopes are now found by

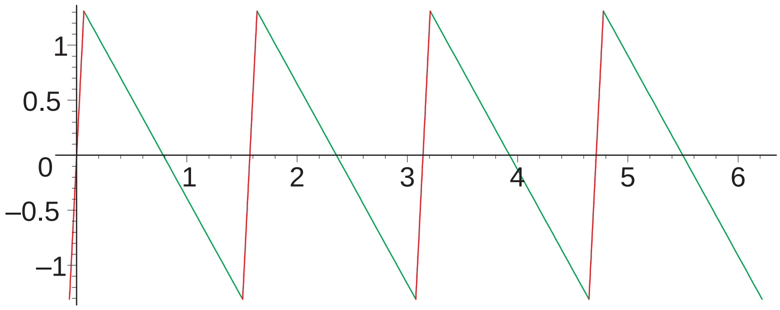


Figure 15: In red (t^+, ϕ) and in green (t^-, ϕ), with $k = 0, \ldots, 3$.

substituting $t = t^\pm$ in (69) or by using (85), see Figure 16 where the curves $\mathbf{y}^+ = \mathbf{X}(\phi, t^+)$ with $k = 0, \ldots, 3$ are plotted in red and the curves $\mathbf{y}^- = \mathbf{X}(\phi, t^-)$ with $k = 0, \ldots, 3$ are plotted in green. Observe, that even though we do not have the full inner part the outer part is now completely determined.

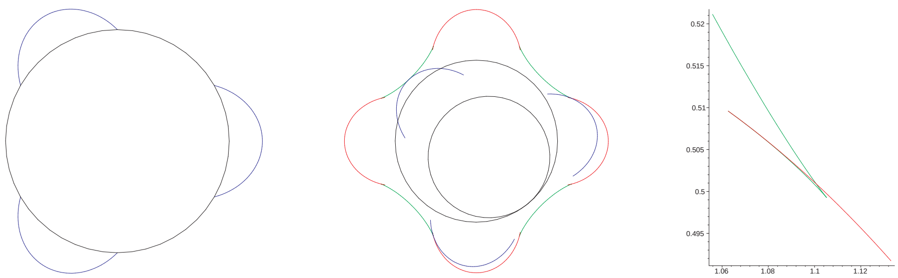


Figure 16: To the left the segment with support function (114). We have also plotted two copies rotated through the angles $2\pi/3$ and $4\pi/3$. In the middle the envelope, in red and green the envelopes using the solutions t^+ and t^- respectively. To the right a blowup that reveals the cusp on one of the envelopes. Notice the different scalings on the axis.

By moving the outer part instead of the inner we can now obtain the full inner part as the envelope of the moving outer part and we can determine the missing segments of the inner part. Consider the part of the envelope given by $\mathbf{y}^-$. Equation (95) becomes

$$t^{-+} = \frac{((n+2)\pi + 2n\phi_0)\phi}{2n(n+1)\phi_0} - \frac{(2\ell(n+1) - 2k + 1)\pi}{n(n+1)}.$$

The missing inner parts are now found by substituting $t = t^{-+}$ in (88) or by using (109), see Figure 18, where the curves $\mathbf{x}^{--} = \mathbf{Y}(\theta, t^{--})$ for $m = 0, 1, 2$ are plotted in blue and the curves $\mathbf{x}^{-+} = \mathbf{Y}(\theta, t^{-+})$ for $m = 0, 1, 2$ are plotted in cyan. In figure 17 we have plotted the support functions h, h^+, h^-, and h^{-+}

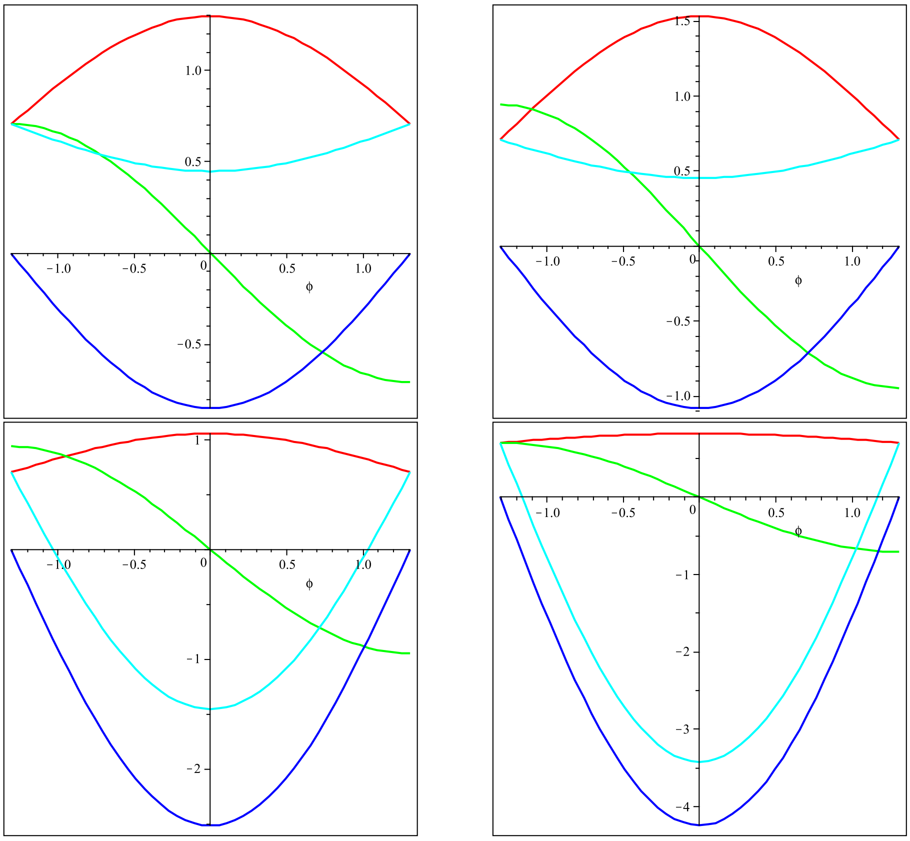


Figure 17: Top left h, h', h'' and $\rho = h + h''$ in red, green, blue, and cyan respectively, top right h^+, bottom left h^-, and bottom right h^{-+}.

together with the first two derivatives and the radius of curvature.

As can be seen from the cyan graphs in Figure 17, this particular construction has a cusp on both the outer and inner part so it does not work exact. If we let $\delta = \pi/2$ then the cusp on the green segments moves to the endpoint, but there appears new cusps at the endpoint of the red, blue, and cyan segments.

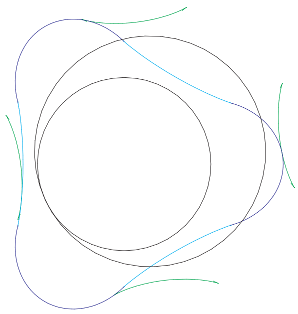


Figure 18: The envelope when the outer part is moving. In blue the old segments, corresponding to the solution t^-. In cyan the missing segments, corresponding to the solution t^+.

In order to obtain a working construction we have to find h such that the first segment is without cusps, i.e, such that $h + h'' \neq 0$. We furthermore want the envelopes to be without cusps, i.e, we want $h^{\pm} + h^{\pm''} \neq 0$ for all segments on the envelope. Finally, the envelope when the outer part is moving has to be without cusps too. Besides the problem with cusp, the example above does not have continuous curvature. It has a jump at the inflexion point, but in contrast to the composed construction in Section 4.2 the jump is finite.

If we want a curvature continuous design then we need a zero for the curvature and that means the radius of curvature $\rho = h + h''$ has a singularity. If the normal direction of the inflexion point is ϕ_0, then the leading term of ρ is proportional to $(\phi - \phi_0)^{-1/2}$, see [2], so the support function h has a term proportional to $(\phi - \phi_0)^{3/2}$.

6 Epi- and Hypo-cycloids on hemispheres

On a sphere of radius R we consider the circle C_w that forms an angle of w radians with the z-axis.

$$C_w(s) = R \begin{bmatrix} \sin w \cos s \\ \sin w \sin s \\ \cos w \end{bmatrix}, \qquad 0 \leq s < 2\pi$$

Furthermore, we need matrices for rotation around the y-axis and the z-axis

$$R_y(v) = \begin{bmatrix} \cos v & 0 & \sin v \\ 0 & 1 & 0 \\ -\sin v & 0 & \cos v \end{bmatrix}$$

$$R_z(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thinking of the parameter t as time, the composition $R_z(at)C_w(s)$ rotates the points $C_w(s)$ on the circle C_w with a speed given by a, such that the image of the full circle is preserved. The composition $R_y(v)R_z(at)C_w(s)$ tilts the rotating circle (around the y-axis) to rotate about an axis with angle v to the z-axes. Finally $f(s, t, a, v, w, R) = R_z(t)R_y(v)R_z(at)C_w(s)$ rotates the axis of the first rotation around the z axis on a circle with angle v to the z-axis. See Figure 19.

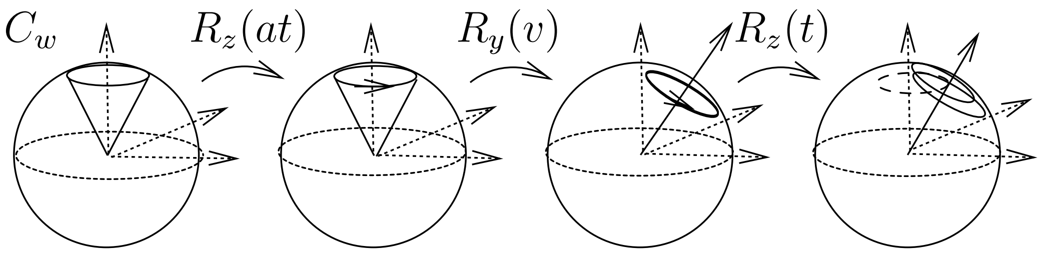


Figure 19: A composition of rotations on a sphere.

The Euclidean radius of the circle C_w is $R \sin w$, giving C_w the circumference $2R\pi \sin w$. This circle should role inside a circle with a circumference that is n times bigger. It then has an Euclidean radius of $nR \sin w$. This outer circle thus makes the angle $\arcsin(n \sin w)$ with the z-axis. The centre of the tilted axes makes thus the angle

$$v = \arcsin(n \sin w) - w$$

with the z-axis. See Figure 20.

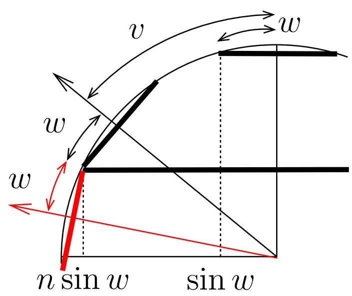


Figure 20: On top, the small circle C_w and parallel with this the n-double circumference circle. The tilt angles of the axis of rotation in both the epi- and hypocycloid cases are illustrated.

For $w < \arcsin(1/n)$, the spherical version of a hypocycloid is given by

$$\mathbf{f}(0, t, -n, \arcsin(n \sin w) - w, w, R)$$
$$= R \begin{bmatrix} \sin(w)\cos(t)\cos(v)\cos(nt) + \sin(w)\sin(t)\sin(nt) + \cos(t)\sin(v)\cos(w)) \\ \sin(w)\sin(t)\cos(v)\cos(nt) - \sin(w)\cos(t)\sin(nt) + \sin(t)\sin(v)\cos(w) \\ \cos(w)\cos(w) - \sin(v)\cos(nt)\sin(w) \end{bmatrix},$$

where $v = \arcsin(n \sin w) - w$. Here the fixed point on the circle C_w is chosen by setting $s = 0$. The tilt of the axis, $v = \arcsin(n \sin w) - w$ is explained on Figure 20. Finally the ratio between the two rotation speeds a is $-n$ as the small inner circle rotates in the opposite direction as its point of contact with the n-times larger outer circle. Similarly, for $w < \arcsin(1/n)$, the spherical version of an epicycloid is given by

$$\mathbf{f}(0, t, n, \arcsin(n \sin w) + w, w, R).$$

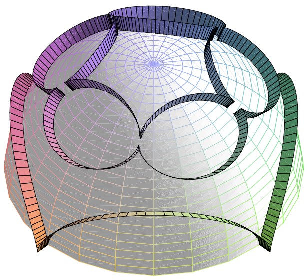


Figure 21: Over a hemisphere is shown: The largest possible hypocycloid with 4 cusps and an epi-/hypo-pair with 5 cusps and almost the same size outer circle.

In the case where $w = \arcsin(1/n)$ the outer circle is the equator and the epi- and hypocycloids are identical. Hence, restricted to a hemisphere we may talk about epi- and hypocycloids, but on the sphere they belong to the same family, which then should be referred to simply as cycloids.

6.1 The motion of a spherical Moineau 'pump' (with one chamber)

To get a circle with Euclidean radius $n \sin w$ to touch a circle with Euclidean radius $(n+1)\sin w$ tangentially, the tilt has to be

$$u = \arcsin\big((n+1)\sin w\big) - \arcsin\big(n \sin w\big).$$

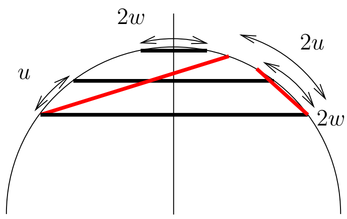


Figure 22: On top, the small circle C_w and parallel with this the 3- and 4-double circumference circles. In red the 3-double circumference circle is tilted by the angle u and C_w is shown at the opposite side where it is to small to seal the gap.

The opening angle at the opposite side is then $2u$ which is greater than than the angle $2w$ covered by the small circle. See Figure 22. The construction is thus not sealed. The gap in Figure 22 is

$$2u - 2w \approx R\Big(n\big(n+1\big)w^3 + \frac{n\big(n+1\big)\big(3n^2+3n+1\big)}{4}w^5 + \dots\Big)$$

For $n = 3$ and $w = \pi/180$ (one degree) the opening is less than 0.4% of w, which seems small. Alternatively, instead of having a slip of 0 (zero) on one side and $2u - 2w$ at the other side, there apparently is space for choosing a tilt of not u, but w giving a slip varying around $u - w$. See Figure 23

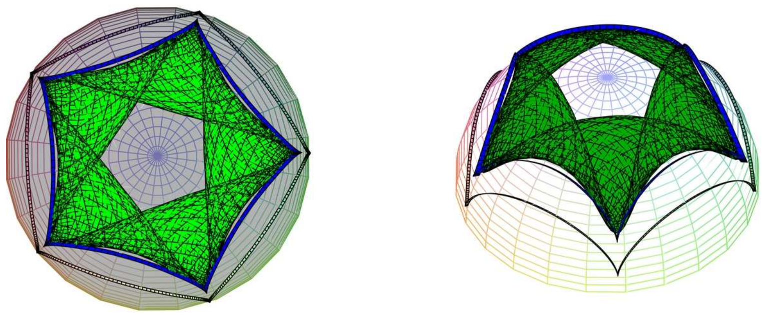


Figure 23: Left: The blue curve is a constant set off of the outer curve by the angle $u - w$ in the direction of the north pole. In green, the motion of the inner part is shown in the case where the tilt of the axis of rotation is $u - w$. Right: The same as to the left hand side, but shown from another angle.

To create the motion of a n-hypocycloid inside a $(n+1)$-hypocycloid, one needs to rotate the n-hypocycloid at speed $-(n+1)/n$, tilt the axis by $u = \arcsin\big((n+1)\sin w\big) - \arcsin\big(n \sin w\big)$ and rotate the tilted axis reversely (speed -1). Using the parameter t as time in this motion and using s to parameterise the moving n-hypocycloid, which then is given by

$$\mathbf{f}(0, s, -n, \arcsin(n \sin w) - w, w, R),$$

we get

$$\mathbf{g}(t,s,n,w,R) = \\ R_z(-t)R_y(u)R_z(-\frac{n+1}{n}t)\mathbf{f}(0,s,-n,\arcsin(n\sin w)-w,w,R).$$

By construction there is one point of contact, which is similar to the moving contact point in the planar case, but the n contact points on the cusps of the inner n-hypocycloid are missing. Another analog to the planar case is to exchange the outer $n+1$-hypocycloid with the curve that the n cusps of the inner n-hypocycloid trace out during the motion. This is shown on Figure 24. The outer curve is the motion of one fixed point on the inner rolling circle given by $\mathbf{g}(t,0,n,w,R)$ for $t \in [0, n2\pi]$. The inner curve is constructed similarly, but for n one smaller. On the left of this figure, note that the great circle segment from the cusp in the front to the centre of the 3-cusp curve goes outside the outer curve. It seems that the curvature of the sphere forces the inner part of the pump to be thinner; in fact so thin that, in the case that the outer curve has a cusp, then the inner part has to have a 'negative' thickness. To the right hand side of this figure it is shown that the cusps of a large hypocycloid do not even trace out a single curve.

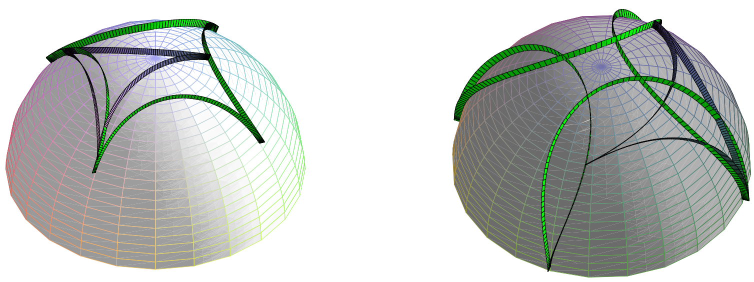


Figure 24: Left: The outer green curve is given by the motion of the cusps of an inner curve, that is by the motion of a fixed point on a radius $3k$ circle rolling on a radius $4k$ circle. Right: As to the left, but for the large circle close the equator.

In partial conclusion, the spherical analogue to cycloids given by rolling a radius n circle inside a radius $n+1$ circle gives the n cusp contact points, but requires an 'negative thickness' of the inner part of the pump. On the other hand the analogue given by rolling an radius r circle on radii n and $n+1$ circles gives the single 'envelope' contact point but does not seal at the n cusp points. However a spherical pump given by a small angle $r = \sin w$ is close to being sealed.

6.2 The area of the spherical pump chambers

In this section we set the radius of the sphere to one. On a radius R sphere the areas simply get multiplied by R^2. On the unit-sphere the Gauss-Bonnet's Theorem for a simple curve states

$$\int_{\Omega\subset\mathbb{S}^2} K\mathrm{d}A + \int_{\partial\Omega} \kappa_g \mathrm{d}s + \sum_{\text{exterior angles}} \epsilon_i = 2\pi,$$

where Ω is the set left of its boundary curve $\partial\Omega$, the Gauss curvature $K = 1$ is one at the unit sphere and κ_g is the geodesic curvature of the boundary curve. As $K = 1$ the first integral equals the area of the set left of the curve. Similarly to the planar case, the geodesic curvature diverges at the cusps and the geodesic curvature is thus hard to integrate numerically to a high precision. Furthermore, when the n-hypocycloid is very small Gauss-Bonnet's Theorem gives a formula of the form $2\pi - 2\pi$ which is bad for the precision of the calculation. Luckily, we can calculate the area directly, and return to Gauss-Bonnet's Theorem to find the integral of the geodesic curvature.

To calculate the area enclosed by a n-hypocycloid

$$\mathbf{r}(t) = \mathbf{f}(0, t, \arcsin(n\sin w) - w, w, 1), \qquad 0 < t < 2\pi,$$

let $\mathbf{e}_z = (0, 0, 1)$ denote the north pole and let

$$\mathbf{h}(\theta, t) = \cos\theta\mathbf{e}_z + \sin\theta\mathbf{r}(t)$$

trace out the desired area when $0 \le \theta \le \pi/2$ and $0 \le t \le 2\pi$. By differentiation, we find

$$\begin{aligned}\frac{\partial\mathbf{h}}{\partial\theta} &= -\sin\theta\mathbf{e}_z + \cos\theta\mathbf{r}(t)\\ \frac{\partial\mathbf{h}}{\partial t} &= \sin\theta\mathbf{r}'(t)\end{aligned}$$

Using that $\mathbf{h}(\theta, t) = \big(x, y, z\big)$ is a normal to the unit sphere at the point $\mathbf{h}(\theta, t)$ we get the area form as

$$\begin{aligned}\mathrm{d}A &= -\mathbf{h}\cdot\Big(\frac{\partial\mathbf{h}}{\partial\theta}\times\frac{\partial\mathbf{h}}{\partial t}\Big)d\theta\mathrm{d}t\\ &= -\big|\cos\theta\mathbf{e}_z + \sin\theta\mathbf{r}(t) \quad -\sin\theta\mathbf{e}_z + \cos\theta\mathbf{r}\big(t\big) \quad \sin\theta\mathbf{r}'\big(t\big)\big|\,\mathrm{d}\theta\mathrm{d}t\\ &= -\Bigg(\big|\cos\theta\mathbf{e}_z \quad \cos\theta\mathbf{r}\big(t\big) \quad \sin\theta\mathbf{r}'\big(t\big)\big|\\ &\qquad\qquad + \big|\sin\theta\mathbf{r}(t) \quad -\sin\theta\mathbf{e}_z \quad \sin\theta\mathbf{r}'\big(t\big)\big|\Bigg)\mathrm{d}\theta\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= -\Big(\cos^2\theta\sin\theta\left|\mathbf{e}_z \quad \mathbf{r}(t) \quad \mathbf{r}'(t)\right| + \sin^2\theta\sin\theta\left|\mathbf{e}_z \quad \mathbf{r}(t) \quad \mathbf{r}'(t)\right|\Big)\mathrm{d}\theta\mathrm{d}t \\
&= -\sin\theta\begin{vmatrix} 0 & x & x' \\ 0 & y & y' \\ 1 & z & z' \end{vmatrix}\mathrm{d}\theta\mathrm{d}t \\
&= \sin\theta(x'y - y'x)\mathrm{d}\theta\mathrm{d}t.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A_n &= \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\mathrm{d}\theta \int_{t=0}^{2\pi}(x'y - y'x)\mathrm{d}t \\
&= \int_{t=0}^{2\pi}(x'y - y'x)\mathrm{d}t \\
&= \pi(\cos w - 1)(\cos w + 1)\Big(-6\cos^4(w)n^2 \\
&\quad + 6n\cos^3(w)\left(1 - n^2 + n^2\cos^2 w\right)^{\frac{1}{2}} - 3\cos^2 w + 3n^2\cos^2 w - 1 + n^2\Big),
\end{aligned}
$$

where the last equality is found using computer algebra.

To calculate the area of the offset, we next assume to have $\mathbf{r}$ parameterised by arc length, denoted s. Hereby $\mathbf{r}'(s)$ is always orthogonal to $\mathbf{r}(s)$ and we may use the positively oriented, orthogonal and normal basis of $\mathbb{R}^3$ given by $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}\times\mathbf{r}'$. We find $\mathbf{r}'' = -\mathbf{r} + \kappa_g\mathbf{r}\times\mathbf{r}'$ where κ_g, the geodesic curvature, is the part of $\mathbf{r}''$ lying in the unit sphere's tangent plane at $\mathbf{r}$. An offset of Θ radians to the right hand side of the curve $\mathbf{r}$ is given by

$$
\mathbf{off}(\theta, s) = \cos\theta\mathbf{r}(s) + \sin\theta\mathbf{r}'(s)\times\mathbf{r}(s).
$$

Differentiation gives:

$$
\begin{aligned}
\frac{\partial\mathbf{off}}{\partial\theta} &= -\sin\theta\mathbf{r}(s) + \cos\theta\mathbf{r}'(s)\times\mathbf{r}(s) \\
\frac{\partial\mathbf{off}}{\partial s} &= \cos\theta\mathbf{r}'(s) + \sin\theta\mathbf{r}''(s)\times\mathbf{r}(s)
\end{aligned}
$$

Similarly to above we get,

$$
\begin{aligned}
\mathrm{d}A &= \left|\mathbf{off} \quad \tfrac{\partial\mathbf{off}}{\partial\theta} \quad \tfrac{\partial\mathbf{off}}{\partial s}\right|\mathrm{d}\theta\mathrm{d}s \\
&= \left|\cos\theta\mathbf{r} + \sin\theta\mathbf{r}'\times\mathbf{r} \quad -\sin\theta\mathbf{r} + \cos\theta\mathbf{r}'\times\mathbf{r} \quad \cos\theta\mathbf{r}' + \sin\theta\mathbf{r}''\times\mathbf{r}\right|\mathrm{d}\theta\mathrm{d}s \\
&= \Big(\cos^3\theta\left|\mathbf{r} \quad \mathbf{r}'\times\mathbf{r} \quad \mathbf{r}'\right| + \cos^2\theta\sin\theta\left|\mathbf{r} \quad \mathbf{r}'\times\mathbf{r} \quad \mathbf{r}''\times\mathbf{r}\right| \\
&\quad - \sin^2\theta\cos\theta\left|\mathbf{r}'\times\mathbf{r} \quad \mathbf{r} \quad \mathbf{r}'\right| - \sin^3\theta\left|\mathbf{r}'\times\mathbf{r} \quad \mathbf{r} \quad \mathbf{r}''\times\mathbf{r}\right|\Big)\mathrm{d}\theta\mathrm{d}s
\end{aligned}
$$

Since $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r} \times \mathbf{r}'$ form an orthogonal and positively oriented basis,

$$\left|\mathbf{r} \quad \mathbf{r}' \times \mathbf{r} \quad \mathbf{r}'\right| = -\left|\mathbf{r} \quad \mathbf{r}' \quad \mathbf{r}' \times \mathbf{r}\right| = 1.$$

Using that identity and that

$$\begin{aligned}
\left|\mathbf{r} \quad \mathbf{r}' \times \mathbf{r} \quad \mathbf{r}'' \times \mathbf{r}\right| &= \left|\mathbf{r} \quad \mathbf{r}' \times \mathbf{r} \quad (-\mathbf{r} + \kappa_g \mathbf{r} \times \mathbf{r}') \times \mathbf{r}\right| \\
&= \left|\mathbf{r} \quad \mathbf{r}' \times \mathbf{r} \quad \kappa_g (\mathbf{r} \times \mathbf{r}') \times \mathbf{r}\right| = \kappa_g \left|\mathbf{r} \quad \mathbf{r}' \times \mathbf{r} \quad (\mathbf{r} \times \mathbf{r}') \times \mathbf{r}\right| \\
&= \kappa_g \left|\mathbf{r} \quad \mathbf{r}' \times \mathbf{r} \quad \mathbf{r}'\right| = \kappa_g,
\end{aligned}$$

we get

$$\begin{aligned}
\mathrm{d}A &= \Big(\cos^3\theta \Big(1\Big) + \cos^2\theta \sin\theta(\kappa_g) - \sin^2\theta\cos\theta(-1) - \sin^3\theta(-\kappa_g)\Big)\, \mathrm{d}\theta \mathrm{d}t \\
&= \left(\cos(\theta) + \sin(\theta)\kappa_g(s)\right) \mathrm{d}\theta \mathrm{d}s.
\end{aligned}$$

Denoting the length of the n-cusp hypocycloid by L, integration gives

$$\begin{aligned}
A_{\mathbf{off}} &= \int_{\theta=0}^{\Theta} \int_{s=0}^{L} \left(\cos(\theta) + \sin(\theta)\kappa_g(s)\right) \mathrm{d}\theta \mathrm{d}s \\
&= L \int_{\theta=0}^{\Theta} \cos(\theta) \mathrm{d}\theta + \int_{\theta=0}^{\Theta} \sin(\theta) \int_{s=0}^{L} \kappa_g(s) \mathrm{d}s \\
&= L \sin\Theta + (1 - \cos\Theta) \int_{s=0}^{L} \kappa_g(s)\, \mathrm{d}s.
\end{aligned}$$

Gauss-Bonnet's Theorem implies that $A_n + \int \kappa_g ds + n\pi = 2\pi$ or

$$\int \kappa_g \mathrm{d}s = \pi(2-n) - A_n.$$

The offset area may thus be written as

$$A_{\mathbf{off}}(n, w) = R^2 \left(L(n, w) \sin\Theta + (1 - \cos\Theta)(\pi(2-n) - A_n)\right),$$

where the length $L\big(n, w\big)$ of the n-cusp hypocycloid for each relevant n has to be found as a function of w by numerical integration.

The offset will form swallowtails near the cusps. The area $A_{\mathbf{off}}$ is calculated with sign, such that a if a point is covered, say 3 times by the offset, 2 times positively and ones negatively, in total is counted as one point of the offset. Finally the area of eventual caps placed on the cusps in the offset, is not included in $A_{\mathbf{off}}$.

7 Conclusion and future work

We have analysed the classical hypocycloid and epicycloid constructions of Moineau pumps. We give closed formulae for the area of the pump chambers as a function of time. We also show that offsets leads to designs with cusps and that they are not mathematically exact. The errors are small and will probably be compensated for by the rubber sealing.

We have also analysed two generalisations suggested by Grundfos. The first consists of alternating arcs of hypo- and epicycloids and here we prove that we have a mathematical exact fit of the outer and inner parts of the design. We also give a closed formula for the area of the pump chambers as a function of time. Even though this design has continuous tangent there are points with infinite curvature, so it might be difficult to machine it. In the second suggestion the inner part is a general hypotrochoid and we determined the outer part of the design, but we have not proved that the parts fits mathematically exact and that the outer part has bounded curvature. Again, an error might be compensated for by the rubber sealing. We have not found a closed expression for the area of pump chambers, but it can be computed numerically. If the design is exact and with bounded curvature then the parts would be easier to machine than the composed design.

We then make an analysis of a general design given by the support function. Assume that the inner part is given by one positively curved segment and one negatively curved segment both repeated n times. Then it seems that the positively curved part alone determines the whole outer part, which then in turn determines the missing segments of the inner part. One has to take care to avoid cusps in the construction and if we want a curvature continuous construction then we need a square-root singularity in the second derivative of the support function. The areas of the pump chambers are given as integrals of relatively simple expressions in the support functions.

We have also considered the generalisation of the hypocycloid construction to the sphere. It turns out that the inner and outer part does not fit mathematically exact. The chambers are not closed but the gaps are small and might be sealed by the rubber sealing. We have also calculated the area of the pump chambers.

As future work we suggest the following

- Finish the analysis of the hypotrochoid construction. Do the parts fit exactly and has the outer part bounded curvature?
- The general approach using support functions should be pursued.
  - The motion is given by a circle rolling in a circle and the radii of the circles are cn and $c(n+1)$ respectively, but what can be said about c when the support function h for the positively curved part is given?

We have assumed that the image of h' should be $[-cn, cn]$, but is that necessary?

– How hard is it to avoid cusps?

– Can we find a class of support functions h that admits many working designs so an optimisation procedure is feasible?

– The support function for the general hypotrochoid should be determined. This would make it easy to check if that design is exact and if it does then it could be a starting point for further designs.

- Another important problem is the spherical design. The general planar approach where the positively curved segment of the inner part is designed first and the rest follows from that can probably be transferred to the sphere. But can the calculation be performed in closed form or is it necessary to resort to numerical calculations?

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