

# Virtual current density in magnetic flow meter

**Problem presented by**

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## Background

Siemens Flow Instruments submits this problem for the 51st European Study Group with Industry, held at the Technical University of Denmark, 16–20 August 2004. The problem relates to performance calculation on magnetic flow meters. A magnetic flow meter measures the velocity of a liquid flowing in a pipe by applying a magnetic field to the flowing liquid and measuring the voltage induced across the liquid as it moves through the magnetic field. This is done by inserting two electrodes into the liquid, one on each side of the pipe. The liquid (typically water) needs to have a minimum conductivity of approximately  $s \approx 10^{-4}$  S/m. If the pipe is made of an electrically conducting material then an insulating liner is used to prevent a short circuit between the pipe and the electrodes. The liner length is typically 3 times the diameter of the pipe. When designing magnetic flow meters it is important to be able to calculate the performance with respect to sensitivity and linearity. That is: The voltage signal generated at a certain flow speed and the variation in signal with flow velocity. This is done by using finite element modelling of the magnetic system, followed by a calculation procedure involving the flow profile of the liquid.

## Theory

When a magnetic field is applied to a moving conductor, a voltage is induced across the conductor. It can be shown that the voltage ( $U$ ) between the two electrodes in a magnetic flow meter is given as:

$$U = \int_V \mathbf{v} \cdot \mathbf{w} dV \quad \text{and} \quad \mathbf{w} = \mathbf{B} \times \mathbf{j} \quad (1)$$

With  $\mathbf{B}$  [T] being the applied magnetic field,  $\mathbf{j}$  [ $\text{m}^{-2}$ ] is a virtual current density resulting from an imaginary current flowing between the two electrodes and  $\mathbf{v}$  [m/s] is the liquid flow velocity. Figure 1 shows an end-view sketch of a magnetic

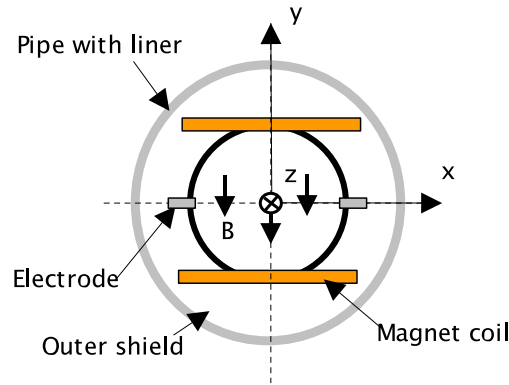


Figure 1: Sketch of a flow meter, end view.

flow meter. The velocity  $\mathbf{v} = [v_x, v_y, v_z]$  can be assumed to have  $u_x = 0$  and  $u_y = 0$ , meaning that the voltage between electrodes can be expressed as:

$$U = \int_0^R \int_0^{2\pi} \int_{-\infty}^{\infty} v_z (B_x j_y - B_y j_x) r dz d\theta dr \quad (2)$$

With  $R$  being the pipe radius. Currently, this calculation is performed by combining the results of two independent finite element calculations. A simulation of the magnetic system combined with a simulation of the electrode system that yields the virtual current.

## Analytical expression for virtual current density in magnetic flow meter

Given the relatively simple geometry of the pipe and electrode system, it seems reasonable to assume that it is possible to derive an analytical expression for the

virtual current so that a rather time consuming finite calculation of the virtual current can be avoided. Figure 2 shows a sketch of the electrode system. Because

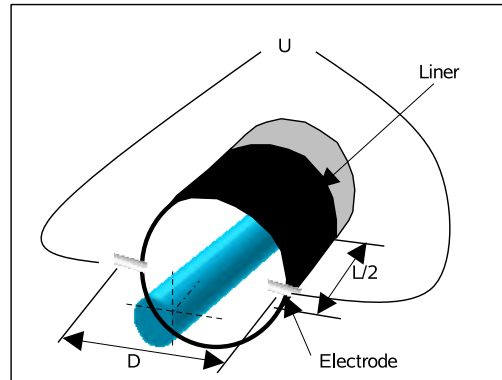


Figure 2: Flow meter electrodes and liner Electrode

of symmetry, the calculation only needs to involve 1/8 of the sensor, 1/4 of the circle and 1/2 the length. The calculation volume is show in figure 3.  $L/2$  is half

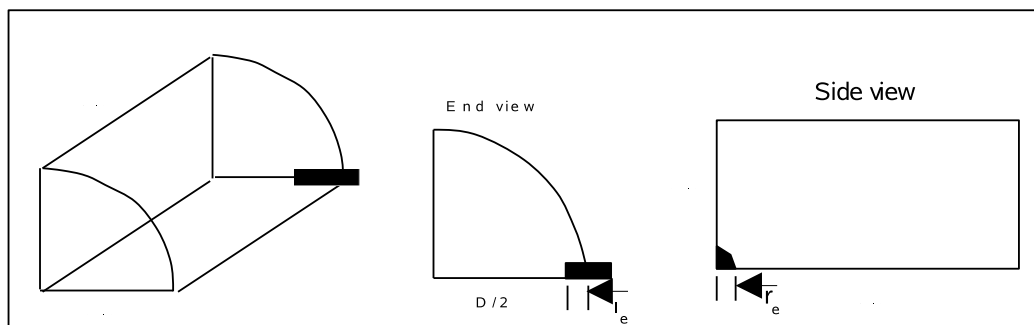


Figure 3: Volume in which the virtual current density must be calculated.

the liner length and  $D/2$  is the pipe radius. The electrode is assumed cylindrical with a radius  $r_e$  and it enters the sensor pipe with a length  $l_e$ . The  $Y-Z$  plane can be set to zero potential ( $V = 0$ ) and the electrode potential is set to  $V = 1V$ .

**The question is now:** Based on the knowledge of tube diameter as well as length and radius of the flow meter electrodes, is it possible to derive an analytical expression for the virtual current density in the tube? That is:  $\mathbf{j} = [j_x, j_y, j_z] = f(x, y, z)$ . This could improve speed and probably also accuracy of the flow meter sensitivity- and linearity calculations. Alternatively, can some rules of scaling be applied so that a finite element calculation on one set of parameters can be transferred to a different set of parameters?

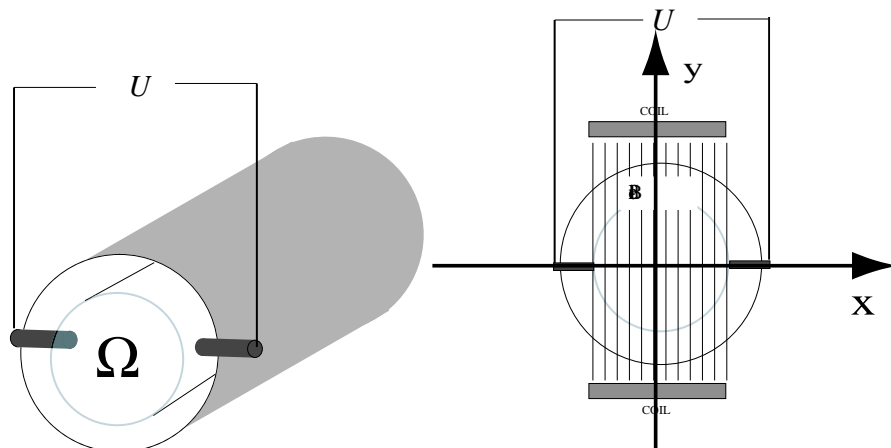
## Study Group contributors

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Report prepared by all of the above.

## 1 The 'Virtual' Current Density $\mathbf{j}$

We begin by some terminology, briefly explaining the formula (1) and the term 'virtual current density'.



We consider the pipe volume  $\Omega$  (see the above figure)) to contain a medium of conductivity  $\sigma$ . We denote the (outward) normal to  $\partial\Omega$  by  $\mathbf{n}$ . We divide the boundary into two disjoint regions: the conducting electrode surfaces ( $\partial\Omega_1$ ) and the non-conducting pipe surface ( $\partial\Omega_2$ ).

The electric and magnetic fields follow Ohm's Law:

$$\mathbf{j} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where  $\mathbf{E}$  is the gradient of a potential, since there is no time dependence, and  $\text{div } \mathbf{j} = 0$  since there is no free charge.

We consider two different experimental situations, I and II, and write down for the quantities  $\mathbf{j}$ ,  $\mathbf{E}$ , and  $\mathbf{B}$  equations that are consistent with each of the two situations. Afterwards we will see that we can relate the two solutions. Combining mathematical identities involving solution fields from separate experimental conditions leads to the use of the term 'virtual current density'.

**Experimental situation (I).** No fluid flow and unit current:  $\mathbf{v}_I = 0$  so that  $\mathbf{j}_I = \sigma \mathbf{E}_I = \sigma \text{grad} \varphi$  on  $\partial\Omega_1$ .

**Experimental situation (II).** Flow meter setup. On the insulating boundary  $\partial\Omega_2$ :  $\mathbf{n} \cdot \mathbf{j}_{II} = 0$ .

We shall use subscripts (I) and (II) to identify the physical quantities arising in each of the situations. Form the product  $\mathbf{j}_I \cdot \mathbf{E}_{II}$  and integrate over the volume  $\Omega$ . From Stoke's theorem we get:

$$\int_{\Omega} \mathbf{j}_I \cdot \mathbf{E}_{II} dV = - \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{j}_I \varphi_{II} dS ,$$

and that

$$\int_{\Omega} \mathbf{j}_{II} \cdot \mathbf{E}_I dV = 0$$

Using Ohm's law (which holds in each situation seperatly): we get

$$\begin{aligned} 0 &= \int_{\Omega} \mathbf{j}_{II} \cdot \mathbf{E}_I dV = \int_{\Omega} \sigma (\mathbf{E}_{II} + \mathbf{v}_{II} \times \mathbf{B}_{II}) \cdot \mathbf{E}_I dV \\ &= \int_{\Omega} \mathbf{j}_I \cdot \mathbf{E}_{II} dV + \int_{\Omega} \mathbf{j}_I \cdot \mathbf{v}_{II} \times \mathbf{B}_{II} dV, \end{aligned}$$

which gives

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{j}_I \varphi_{II} dS = \int_{\Omega} \mathbf{j}_I \cdot \mathbf{v}_{II} \times \mathbf{B}_{II} dV \quad (3)$$

If we now assume that in situation (I) a unit current flows to and from each electrode and nowhere else on the boundary, we have that

$$\mathbf{n} \cdot \mathbf{j}_I \approx \begin{cases} 1/A & \text{on electrode 1} \\ -1/A & \text{on electrode 2} \\ 0 & \text{elsewhere} \end{cases}$$

where  $A$  denotes the electrode area. Hence, the left hand of (3) equals approximately  $\Delta\varphi$ , the potential difference in situation (II). Thus, dropping the subscripts (I) and (II), we obtain

$$U \equiv \Delta\varphi_{II} = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} \times \mathbf{B} dV$$

which is the flow meter equation (1). Thus, there is nothing 'virtual' about  $\mathbf{j}$ , it is simply a solution for the current density in a different setup which can be combined with the actual fields to give a compact expression for the measured potential difference  $U$ .

## 2 The boundary conditions

We model the flow meter as a cylinder and the electrodes as symmetrically placed patches that are intervals in  $z$  and in  $\theta$  respectively. See figure 4.

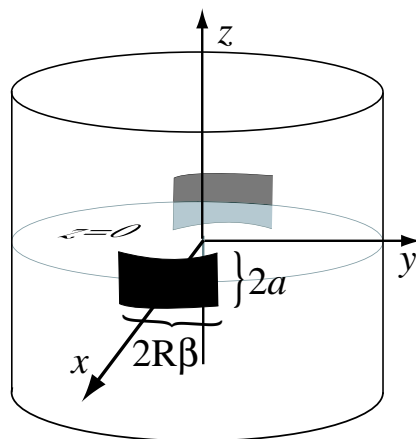


Figure 4: The shape and placement of the electrodes

Finding the potential  $\varphi$  for the virtual current density amounts to solving the Laplace equation with mixed boundary conditions.

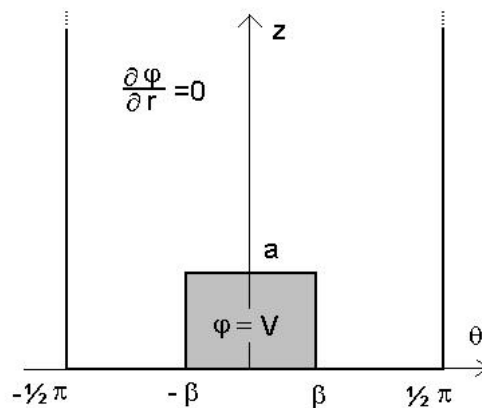


Figure 5: Mixed boundary conditions

Figure 5 shows the mixed boundary conditions for  $r = R$ , where the shaded area is half of the electrode area  $A = (2\beta R) \times (2a)$ . The potential  $\varphi(r, \theta, z)$  must

satisfy the conditions

$$\begin{aligned} \varphi(r, \theta, z) &= V, & (r, \theta, z) \in \partial D_1, & \text{electrode} \\ \frac{\partial \varphi(r, \theta, z)}{\partial r} &= 0, & (r, \theta, z) \in \partial D_2, & \text{insulator} \end{aligned} \quad (4)$$

This mixed problem involving both a Neumann and a Dirichlet condition is nontrivial. Using a Wiener-Hopf technique, see [2], such a mixed problem can be solved in certain cases. Here we shall employ an iterative method described in [3]. This method is described in the next section.

### 3 The iterative solution

#### 3.1 First order solution

For the lowest order approximation  $\varphi^{(1)}$  to the potential we assume that the current density is a constant across the electrode surfaces. This assumption gives the boundary condition

$$\frac{\partial \varphi^{(1)}(R, \theta, z)}{\partial r} = \begin{cases} I^{(1)}/(\sigma A), & (r, \theta, z) \in \partial D_1, \text{ electrode,} \\ 0, & (r, \theta, z) \in \partial D_2, \text{ insulator,} \end{cases} \quad (5)$$

where  $I^{(1)}$  is a constant of order unity representing the first order approximation of the current running through the electrode.  $A = 4\beta R a$  is the total area of one of the electrodes and  $\sigma$  is the conductivity.

#### 3.2 Second order solution

The first order solution  $\varphi^{(1)}(r, \theta, z)$  given by (13) satisfy the Neumann conditions given in (5). The idea is that the second order solution must satisfy the Dirichlet condition

$$\varphi^{(2)}(R, \theta, z) = \begin{cases} V, & (R, \theta, z) \in \partial D_1, \text{ electrode,} \\ \varphi^{(1)}(R, \theta, z), & (R, \theta, z) \in \partial D_2, \text{ insulator,} \end{cases} \quad (6)$$

where  $V$  is a constant. This solution will give the right potential at the electrodes but may not satisfy the Neumann condition on the insulator. Nevertheless as we shall see the approximation  $\varphi^{(1)}$  is already quite close to the exact solution  $\varphi$ .

We now turn to construction of the first approximation  $\varphi^{(1)}$ . We will derive an analytical expression for  $\varphi^{(1)}$  in two separate ways; first in section 4 from a separation of variables solution of the Laplace equation, then, in section 5 using a Green's function approach.

## 4 Separation of variables solution for $\varphi^{(1)}$

### 4.1 Cylindrical coordinates: $\varphi(r, \theta, z)$

We seek a solution for the Laplace equation  $\nabla^2\varphi = 0$  which in cylindrical coordinates is:

$$\nabla^2\varphi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial\varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0. \quad (7)$$

To solve this equation we write a solution as a product of three ordinary functions

$$\varphi(r, \theta, z) = f(r) g(\theta) h(z)$$

Substituting this expression of  $\varphi$  into equation (7) we obtain three separate ordinary differential equations

$$\begin{aligned} r(rf'(r))' - (m^2 + k^2r^2)f(r) &= 0, & \text{Bessel's modified equation,} \\ g''(\theta) + m^2g(\theta) &= 0, & \text{Harmonic oscillator equation,} \\ h''(z) + k^2h(z) &= 0, & \text{Harmonic oscillator equation.} \end{aligned}$$

The general solutions to these equations are:

$$\begin{aligned} \varphi(r, \theta, z) &= [A_m(k)I_m(kr) + B_m(k)K_m(kr)] \\ &\times [C_m(k)\cos(kz) + D_m(k)\sin(kz)][E_m(k)\cos(m\theta) + F_m(k)\sin(m\theta)] \end{aligned} \quad (8)$$

where  $I_m(x)$  and  $K_m(x)$  are the modified Bessel functions of 1<sup>st</sup> and 2<sup>nd</sup> kind.

### 4.2 Symmetries and symmetry planes

The symmetries in the present geometry (compare figure 4) are:

**Periodicity in  $\theta$  :**  $m \in \mathbb{N}$

**Symmetry in  $xz$ -plane:**  $\varphi(r, \theta, z) = \varphi(r, -\theta, z) \Rightarrow F_m(k) = 0$

**Anti-symmetry in  $yz$ -plane:**  $\varphi(r, \theta, z) = -\varphi(r, \pi - \theta, z) \Rightarrow m$  is odd.

**Symmetry in the  $xy$ -plane:**  $\varphi(r, \theta, z) = \varphi(r, \theta, -z) \Rightarrow D_m(k) = 0$

Introducing these symmetries into the solution (8) and using the principle of superposition we finally get the following form of the general solution to (7)

$$\varphi(r, \theta, z) = \sum_{m \text{ odd}} \left[ \int_0^\infty A_m(k) I_m(kr) \cos(kz) dk \right] \cos(m\theta). \quad (9)$$

We shall denote by  $m'$  a sum over odd values for  $m$  only. Equation(5) leads to the double Fourier expansion:

$$\sum_{m'} \left[ \int_0^\infty A_m^{(1)}(k) I_m'(kR) k \cos(kz) dk \right] \cos(m\theta) = \begin{cases} I^{(1)}/(\sigma A), & |\theta| \leq \beta, \quad 0 \leq z \leq a, \\ 0 & \text{elsewhere,} \end{cases} \quad (10)$$

and involves both a Fourier series

$$g(\theta) = \sum_{m'} a_m \cos(m\theta), \quad a_m = \frac{2}{\pi} \int_0^\pi g(\theta) \cos(m\theta) d\theta, \quad m \text{ odd} \quad (11)$$

and a Fourier cosine-transform

$$h(z) = \int_0^\infty H(k) \cos(kz) dk, \quad H(k) = \frac{2}{\pi} \int_0^\infty h(z) \cos(kz) dz \quad (12)$$

Applying the two inverse transformations (11) and (12) to equation (10), we find

$$A_m^{(1)}(k) I_m'(kR) k = 2 \frac{2}{\pi} \frac{2}{\pi} \frac{I^{(1)}}{\sigma A} \int_0^a \cos(kz) dz \int_0^\beta \cos(m\theta) d\theta, \quad m \text{ odd}$$

which gives

$$A_m^{(1)}(k) = \frac{2I^{(1)}}{\sigma a R \beta \pi^2} \frac{\sin(ka)}{I_m'(kR) k^2} \frac{\sin(m\beta)}{m}, \quad m \text{ odd}$$

Finally we then have the expression

$$\varphi^{(1)}(r, \theta, z) = \frac{2I^{(1)}}{\sigma a R \beta \pi^2} \sum_{m'} \left[ \int_0^\infty \frac{I_{m'}(kr) \sin(ka) \cos(kz)}{I_{m'}'(kR) k^2} dk \right] \frac{\sin(m'\beta)}{m} \cos(m'\theta) \quad (13)$$

## 5 Green's function method

An alternative (though closely related) method is to use a Green's function [1]:

$$\varphi^{(1)}(\mathbf{x}) = \frac{1}{4\pi\epsilon} \int_V \rho_f(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d^3x' + \frac{1}{4\pi} \int_S \left[ G(\mathbf{x}, \mathbf{x}') \frac{\partial \varphi}{\partial n'} - \varphi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] dS', \quad (14)$$

where  $G(\mathbf{x}, \mathbf{x}')$  called Green's function is required to satisfy the following inhomogeneous differential equation:

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{x}') = -\frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\theta - \theta') \delta(z - z'). \quad (15)$$

Here,  $\epsilon$  and  $\rho_f$  are the permittivity and the charge density, respectively, and  $\rho, \theta, z$  are the usual cylindrical coordinates. Since no electrical sources appear inside the cylinder surface in our problem, the expression for the potential  $\varphi$  simplifies to:

$$\varphi(\mathbf{x}) = \frac{1}{4\pi} \int_S \left[ G(\mathbf{x}, \mathbf{x}') \frac{\partial \varphi}{\partial n'} - \varphi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] dS'. \quad (16)$$

Since we are solving for the approximation  $\varphi^{(1)}$  we choose our Green's function in such a way that it satisfies a Neumann conditions everywhere on the cylinder surface, i.e.,

$$\left. \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right|_S = 0, \quad (17)$$

as Equation (16) then simplifies to:

$$\varphi(\mathbf{x}) = \frac{1}{4\pi} \int_S G(\mathbf{x}, \mathbf{x}') \frac{\partial \varphi}{\partial n'} dS'. \quad (18)$$

Using the following expressions:

$$\delta(z - z') = \frac{1}{\pi} \int_0^\infty \cos[k(z - z')] dk,$$

$$\delta(\theta - \theta') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\theta - \theta')}, \quad (19)$$

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty} \int_0^\infty e^{im(\theta - \theta')} \cos[k(z - z')] g_m(k, \rho, \rho') dk, \quad (20)$$

then equation 15 is formally satisfied if  $g_m(k, \rho, \rho')$  satisfies

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dg_m}{d\rho} \right) - \left( k^2 + \frac{m^2}{\rho^2} \right) g_m = -\frac{4\pi}{\rho} \delta(\rho - \rho'). \quad (21)$$

When  $\rho \neq \rho'$ , this equation is Bessel's modified equation which has the linearly independent solutions  $I_m(k\rho)$  and  $K_m(k\rho)$ . Since  $g_m(k, \rho, \rho')$  is bounded as  $\rho$  tends to zero, we may write:

$$\begin{aligned} g_m(k, \rho, \rho') &= A(\rho') I_m(k\rho), & 0 \leq \rho < \rho', \\ g_m(k, \rho, \rho') &= B_1(\rho') I_m(k\rho) + B_2(\rho') K_m(k\rho), & \rho' < \rho \leq R. \end{aligned} \quad (22)$$

Here  $R$  is the cylinder radius. The functions  $A, B_1, B_2$  are, at this point, unspecified functions of  $\rho'$ . Next, imposing Equation (17), symmetry in  $g_m$ :  $g_m(k, \rho, \rho') = g_m(k, \rho', \rho)$ , and employing the Wronskian relation

$$W[I_m(x), K_m(x)] = -\frac{1}{x}, \quad (23)$$

allows us to write for  $g_m$ :

$$g_m(k, \rho, \rho') = -4\pi \frac{K'_m(kR)}{I'_m(kR)} \left( I_m(k\rho_{<}) \left[ I_m(k\rho_{>}) - \frac{I'_m(kR)}{K'_m(kR)} K_m(k\rho_{>}) \right] \right), \quad (24)$$

where  $\rho_{>} = \max(\rho, \rho')$  and  $\rho_{<} = \min(\rho, \rho')$ . Note that Equation (24) simplifies to:

$$g_m(k, \rho, R) = \frac{4\pi}{kR} \frac{I_m(k\rho)}{I'_m(kR)},$$

when evaluated on the cylinder surface:  $\rho' = R$  [refer to the surface area integration in Equation (18)].

Carrying out the integration in Equation (18) over the appropriate coordinate ranges for the electrodes yields:

$$\begin{aligned} \varphi^{(1)}(r, \theta, z) &= \frac{2I^{(1)}}{\sigma a R \beta \pi^2} \sum_{m'} \left[ \int_0^\infty \frac{I_{m'}(kr) \sin(ka) \cos(kz)}{I'_{m'}(kR) k^2} dk \right] \frac{\sin(m'\beta)}{m} \cos(m'\theta) \end{aligned} \quad (25)$$

where use has been made of Equations (20), (24) and the fact that:

$$\int_{-\beta}^{\beta} e^{-im\theta'} d\theta' = \frac{2}{m} \sin(m\beta), \quad (26)$$

$$\int_{\pi-\beta}^{\pi+\beta} e^{-im\theta'} d\theta' = \frac{2(-1)^m}{m} \sin(m\beta), \quad (27)$$

$$\int_{-a}^a \cos(k(z-z')) dz' = -\frac{1}{k} [\sin(k(z-a)) - \sin(k(z+a))]. \quad (28)$$

This expression is identical to that of equation (13).

Having now derived the first order expression for  $\varphi^{(1)}$  in two different ways we now examine the analytical expression numerically.

## 6 A Numerical example

We consider the values  $\sigma = 1$ ,  $I^{(1)} = 1$ ,  $a = \pi/10$ ,  $\beta = \pi/10$ , and  $R = 1$ .

We substitute the above values into the first order solution given in (13). The numerics are done in MAPLE 9.03. To minimize the CPU-time the Bessel functions are approximated by MacLaurin series with 21 terms and the sum of the series of  $\varphi^{(1)}$  is truncated such that  $m$  takes the values  $\{1, 3, 5, \dots, 19\}$ . The integration over  $k$  is performed in the interval  $[0, 30]$ . The results of the calculations are collected in the figures below.

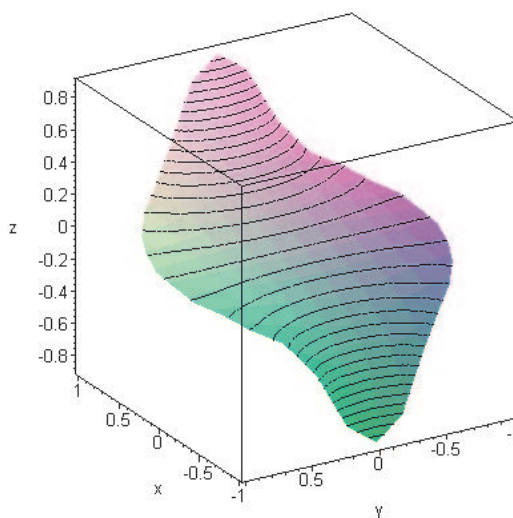


Figure 6: The values of the potential  $\varphi^{(1)}$  in the plane  $z = 0$ .

## 7 Second order solution

Consider again the defining equation (6) for the second approximation  $\varphi^{(2)}$ . The coefficient  $A_m^{(2)}(k)$  can be determined from

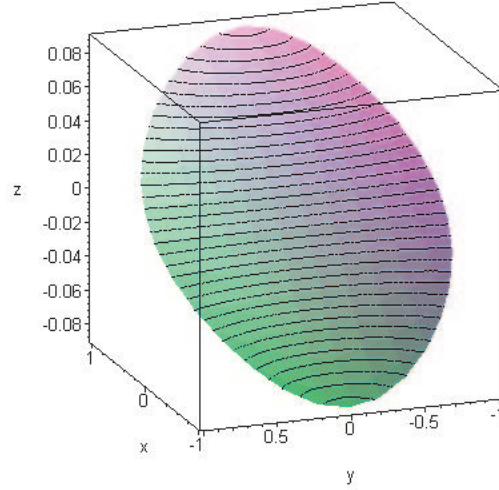


Figure 7: The values of the potential  $\varphi^{(1)}$  in the plane  $z = 1$ . Notice the scale.

$$A_m^{(2)}(k)I_m(kR) = A_m^{(1)}(k)I_m(kR) - \frac{4}{\pi^2} \int_0^a \cos(kz) dz \int_0^\beta \varphi^{(1)}(R, \theta, z) \cos(m\theta) d\theta + \frac{8V \sin(ka) \sin(m\beta)}{\pi^2 k m}, \quad m \in \{1, 3, 5, \dots\}. \quad (29)$$

The double integral in (29) can be simplified as

$$\int_0^a \cos(kz) dz \int_0^\beta \varphi^{(1)}(R, \theta, z) \cos(m\theta) d\theta = \frac{2I^{(1)} \sin(m\beta) \pi}{\sigma a R \beta \pi^2 m} \frac{\pi}{2} \times \int_0^\infty \frac{I_m(uR) \sin(ua) [k \sin(ak) \cos(au) - u \cos(ak) \sin(au)]}{I'_m(uR) u^2 (k^2 - u^2)} du.$$

The total current  $I^{(2)}$  can be determined by

$$I^{(2)} = 4\sigma R \int_0^a dz \int_0^\beta \frac{\partial \varphi^{(2)}}{\partial r}(R, \theta, z) d\theta.$$

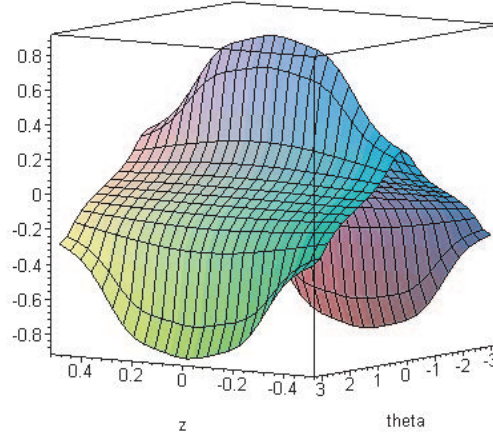


Figure 8: The values of the potential  $\varphi^{(1)}$  at the boundary  $R = 1$

## 8 The weight factor

In this section we calculate an analytical expressions for the response of the flow meter. Start with the flow meter equation

$$U = \int_{\Omega} \mathbf{j} \cdot \mathbf{v} \times \mathbf{B} dV \quad (30)$$

where  $\mathbf{j}$  is the current density for a unit current, that is,

$$\mathbf{j} = -\frac{\sigma}{I} \left[ \frac{\partial \varphi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \vec{e}_\theta + \frac{\partial \varphi}{\partial z} \vec{e}_z \right]. \quad (31)$$

The  $\mathbf{B}$ -field is assumed homogeneous and parallel to the  $y$ -axis

$$\mathbf{B} = B \vec{e}_y = B \sin(\theta) \vec{e}_r + B \cos(\theta) \vec{e}_\theta. \quad (32)$$

Further we assume that the fluid velocity field is axisymmetric with respect to the  $z$ -axis such that

$$\mathbf{v} = v(r) \vec{e}_z. \quad (33)$$

Substituting (31), (32) and (33) into (30) we get the expression for  $\varphi$

$$U = \frac{\sigma B}{I} \int_0^R v(r) r dr \int_0^{2\pi} W(r, \theta) d\theta, \quad (34)$$

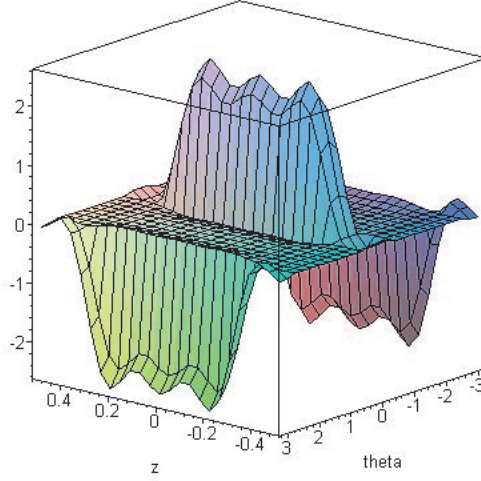


Figure 9: The current density  $J^{(1)}$  at the boundary  $R = 1$

where the weight factor  $W(r, \theta)$  is

$$W(r, \theta) = 2 \int_0^\infty \left( \cos(\theta) \frac{\partial \varphi}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial \varphi}{\partial \theta} \right) dz. \quad (35)$$

From the Fourier cosine-transform (11) we have

$$\int_0^\infty h(z) dz = \frac{\pi}{2} H(0), \quad (36)$$

and utilizing (36) we find for (35)

$$W(r, \theta) = \pi \sum_m^\infty \left( \cos(\theta) \cos(m\theta) B_m(r) + m \sin(\theta) \sin(m\theta) \frac{C_m(r)}{r} \right), \quad (37)$$

where

$$B_m(r) = \lim_{k \rightarrow 0} A_m(k) I_m'(kr) k, \quad C_m(r) = \lim_{k \rightarrow 0} A_m(k) I_m(kr). \quad (38)$$

For  $U$  given by (34) we finally find

$$U = \frac{\pi^2 \sigma B}{I} \int_0^R v(r) [r B_1(r) + C_1(r)] dr. \quad (39)$$

## 8.1 First order solution

For the first order solution  $\varphi^{(1)}(r, \theta, z)$  given in (13) we get

$$rB_1(r) + C_1(r) = \frac{4I^{(1)}}{\sigma\pi^2R} \frac{\sin(\beta)}{\beta} r. \quad (40)$$

This gives for  $\varphi^{(1)}$

$$\varphi^{(1)} = 4B R v_{\text{ave}} \frac{\sin(\beta)}{\beta}, \quad (41)$$

and the the sensitivity of the flow meter can be found as

$$S^{(1)} = \frac{U^{(1)}}{2RB v_{\text{ave}}} = \frac{2 \sin(\beta)}{\beta}. \quad (42)$$

Notice that the sensitivity is not increased by taking larger electrode area (i.e., larger  $\beta$ ).

## 9 Summary

We have demonstrated that in a simple geometry where the cylindrical coordinate expression for the electrodes are intervals in  $\theta$  and  $z$  it is indeed possible to systematically compute a series of analytical approximations to the 'virtual' current density. We give a detailed derivation of the first order approximation and indicate how to proceed with higher order approximations. Numerics indicate that even the first order approximation is fairly accurate.

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